MAA OMINATI DEGREE COLLEGE

Sub. – Calculus

Class – BSC / B.A 2nd sem.

Session – 2024–25

Mohit Tanwar

L'Hôpital's Rule and Indeterminate Forms

CREATED BY TYNAN LAZARUS

October 29, 2018

Now that we have the power of the derivative, we can use it as a way to compute limits that we didn't have the ability to understand before. Early on, we could compute limits of rational functions quite easily. However, we couldn't deal with mixing a ratio of different kinds of functions, like a polynomial and an exponential. L'Hôpital's Rule allows us to evaluate these kinds of limits without much effort. It also allows us to deal with different indeterminate forms. We will see through some examples just how weird ∞ can act and why these indeterminate forms bring about contradictions in our intuition.

1.1 The Definition

Theorem (L'Hôpital's Rule): Let $f(x)$ and $g(x)$ be differentiable on an interval I containing a, and that $g'(a) \neq 0$ on I for $x \neq a$. Suppose that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{or} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Then as long as the limits exist, we have that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

There is an analogous version for when a is ∞ or $-\infty$. What this theorem essentially says is that if you tried to compute the limit of a ratio of functions, but you get the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then you can compute the limit of the ratio of the derivatives of those functions instead. However, take caution that it is not necessarily a short cut. When encountering limits that we have seen before, it may be faster to use a different technique than L'Hôpital's Rule. Also note that we are ***not*** taking a quotient rule. We just take the derivatives of the top and the bottom of the fraction and leave them there.

1.2 How it Works

Before we could compute the derivative of $\sin(x)$ or $\cos(x)$, we had to figure out two trig limits. We found that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$ by using some geometry tricks with sectors and whatnot. Now that we know how to compute derivatives, we can use L'Hôpital's Rule to check that this is correct.

In order to use L'Hôpital's Rule, we need to check that it is in the right form and that we get one of the indeterminate forms required. As usual with limits, we attempt to just plug in the value and see if we get a number. If we did get a real number, then we are done. Here we can see that if we try to plug in $x = 0$ in the limit, we get that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0}$, which is an indeterminate form. Therefore, we can apply L'Hôpital's Rule. Whenever we do so, we will use a "$\overset{\text{L'H}}{=}$" to denote that we have used the rule and "=" to denote our usual simplification. So, applying L'Hôpital's Rule, we get $\lim\limits_{x\to 0} \frac{\sin(x)}{x} \overset{\text{L'H}}{=} \lim\limits_{x\to 0} \frac{\cos(x)}{1}$. However, this second expression is a limit of a continuous function, so we can just plug $x = 0$ and get that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} \overset{\text{L'H}}{=} \lim\limits_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$, verifying what we already know to be true.

We can do the same with our other trig limit, $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$. First, we have to check that L'Hôpital's Rule even applies. If we tried to plug in $x = 0$ we would get $\frac{\cos(0)-1}{0} = \frac{0}{0}$, which is one of our indeterminate forms. L'Hôpital's Rule does apply to this form, so we get that $\lim_{x\to 0} \frac{\cos(x)-1}{x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\sin(x)}{1} = -\sin(0) = 0$.

1.3 Examples with Indeterminate Forms

1.3.1 $\frac{0}{0}$ Form

Question: Why is $\frac{0}{0}$ indeterminate? In general, $\frac{0}{\text{stuff}} = 0$, and $\frac{\text{stuff}}{0}$ acts like ∞. So the top pulls the limit down towards zero, and the bottom pulls it up to infinity. So who wins?

- Let's say we want to compute $\lim_{x\to 2} \frac{x-2}{x^2-4}$. We can see that if we try to plug in $x = 2$, we get $\frac{0}{0}$. Therefore we can apply L'Hôpital's Rule to get

$$\lim_{x\to 2} \frac{x-2}{x^2-4} \overset{\text{L'H}}{=} \lim_{x\to 2} \frac{1}{2x} = \frac{1}{2(2)} = \frac{1}{4}.$$

But this is a limit that we could've computed in the first week of the course; we don't even need the relative canon that is L'Hôpital to swat this little limit. Earlier we would've just factored the bottom and gotten

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$$

Either way is just as quick because this is a simple limit.

- For a more interesting example, let's try to compute $\lim_{x\to 0} \frac{\ln(\sec(x))}{3x^2}$. We see a limit, so our first instinct is to plug in the limiting value $x = 0$. When we do this, we get $\frac{\ln(\sec(0))}{3(0)^2} = \frac{\ln(1)}{0} = \frac{0}{0}$. This is one of the indeterminate forms that L'Hôpital's Rule can help us with. So, we use it to get

$$\lim_{x\to 0} \frac{\ln(\sec(x))}{3x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\sec(x)} \cdot \sec(x)\tan(x)}{6x} = \lim_{x\to 0} \frac{\tan(x)}{6x}.$$

Trying to take this limit also results in $\frac{0}{0}$. So did L'Hôpital fail us? Not quite. All L'Hôpital tells us is that the limit of the original ratio is that same as the limit of the ratio of the derivatives. We got $\frac{0}{0}$, which is what L'Hôpital's Rule is designed for, so let's use it again! Thus, we get

$$\lim_{x\to 0} \frac{\ln(\sec(x))}{3x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\tan(x)}{6x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\sec^2(x)}{6} = \frac{1}{6}.$$

Therefore, our original limit has a value of 1/6. This problem shows us that you may need to use L'Hôpital's Rule multiple times before we get an answer. However, we do need to check that we are in the correct indeterminate form each time before we can apply it.

Unit-1

1.3.3 $0 \cdot \infty$ Form

Question: Why is $0 \cdot \infty$ indeterminate? Usually $0 \cdot (\text{stuff}) = 0$ and $(\text{stuff}) \cdot \infty = \infty$. So one piece tries to pull the limit down to zero, and the other tries to pull it up to ∞. Does one side win? Or do they sort of balance each other out and we get an answer of something like 7?

- As a first example, let's compute $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$. We can see that as x goes off to infinity, $1/x$ goes to zero and $\sin(0) = 0$. So we have the form $\infty \cdot 0$. However, this isn't a form that L'Hôpital's Rule can be used on. In order to determine what value the limit approaches we have to first put it in the correct form. The trick that we will use is a way to rewrite x. Recall that $x = \frac{1}{\frac{1}{x}}$. Using this, we can rewrite the given limit as follows,

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}.$$

  Now if we look at this limit and consider x tending to ∞, we see the top approaches 0 and the bottom also approaches 0. Thus, we are now in the correct form for L'Hôpital's Rule. Thus, we get

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1.$$

  We can see that sometimes we'll need to do some manipulation of the terms in the limit before we can use L'Hôpital's Rule.

- Now let's compute $\lim_{x\to 0^+} x^3 \ln(x)$. As x approaches zero, we can see that we get the form $0 \cdot -\infty$. Following the same trick as last time, we can compute that the value of this limit is

$$\lim_{x\to 0^+} x^3\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-3}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-3x^{-4}} = \lim_{x\to 0^+} \frac{x^3}{-3} = 0.$$

- For a slightly trickier example, consider $\lim_{t\to\frac{\pi}{2}^-} \tan(t)\sin\left(t - \frac{\pi}{2}\right)$. We can see that this is of the $\infty \cdot 0$ type, but we can't use the same trick as last time. When in doubt with trigonometric functions, turn everything back into sines and cosines. Here, we get

$$\lim_{t\to\frac{\pi}{2}^-} \tan(t)\sin\left(t-\frac{\pi}{2}\right) = \lim_{t\to\frac{\pi}{2}^-} \frac{\sin(t)}{\cos(t)}\sin\left(t-\frac{\pi}{2}\right),$$

  which is in the $\frac{0}{0}$ form now. Thus, we can now use L'Hôpital's Rule. Therefore, we calculate

$$\begin{aligned}\lim_{t\to\frac{\pi}{2}^-} \tan(t)\sin\left(t-\frac{\pi}{2}\right) &= \lim_{t\to\frac{\pi}{2}^-} \frac{\sin(t)\sin\left(t-\frac{\pi}{2}\right)}{\cos(t)} \\ &\overset{\text{L'H}}{=} \lim_{t\to\frac{\pi}{2}^-} \frac{\cos(t)\sin\left(t-\frac{\pi}{2}\right)+\sin(t)\cos\left(t-\frac{\pi}{2}\right)}{-\sin(t)} \\ &= \frac{\cos\left(\frac{\pi}{2}\right)\sin(0)+\sin\left(\frac{\pi}{2}\right)\cos(0)}{-\sin\left(\frac{\pi}{2}\right)} = -1.\end{aligned}$$

1.3.2 $\frac{\infty}{\infty}$ Form

Question: Why is $\frac{\infty}{\infty}$ indeterminate? Usually $\frac{\infty}{\text{stuff}}$ acts like ∞ and $\frac{\text{stuff}}{\infty}$ goes to 0. So the top pulls the limit up to infinity and the bottom tries to pull it down to 0. So who wins?

- Consider the following limit, $\lim_{x\to\infty}\frac{2x^2}{e^{3x}}$. Since this ratio is of a polynomial and an exponential function, we can't solve it with any of the usual techniques from earlier in the course. We can see that if we could plug in larger and larger values that $2x^2$ diverges up to infinity, and so does e^{3x}. Thus, this limit looks like $\frac{\infty}{\infty}$, which L'Hôpital's Rule can handle. We get that

$$\lim_{x\to\infty}\frac{2x^2}{e^{3x}} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{4x}{3e^{3x}}.$$

But we can see that this second limit is also $\frac{\infty}{\infty}$, so we can apply L'Hôpital's Rule again to get

$$\lim_{x\to\infty}\frac{2x^2}{e^{3x}} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{4x}{3e^{3x}} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{4}{9e^{3x}} = 0.$$

Remember that we can apply L'Hôpital's Rule as many times as is needed. However, this can backfire.

- Consider $\lim_{x\to-\infty}\frac{x^{67}-3x^{40}+x+1}{x^{12}+2x^{64}-x^2-5}$. If we try the limit, we see that we get $\frac{-\infty}{\infty}$. We could blindly try to use L'Hôpital's Rule, but when we do that we only reduce the degrees of the numerator and denominator by one. We would still be left with large powers on the top and the bottom, and it would still be some sort of $\frac{\pm\infty}{\pm\infty}$. In fact, we would have to do L'Hôpital's Rule 64 times before we get an answer that is not in an indeterminate form!

This problem can be much more easily done with our old technique. We can see that we get

$$\lim_{x\to-\infty}\frac{x^{67}-3x^{40}+x+1}{x^{12}+2x^{64}-x^2-5}\cdot\frac{1/x^{64}}{1/x^{64}} = \lim_{x\to-\infty}\frac{x^3-\frac{3}{x^{24}}+\frac{1}{x^{63}}+\frac{1}{x^{64}}}{\frac{1}{x^{52}}+2-\frac{1}{x^{62}}-\frac{5}{x^{64}}} = -\infty.$$

- Now let's compute $\lim_{x\to0^+}\frac{\ln(e^x-1)}{\ln(x)}$. When we try to plug in $x=0$ (or rather, smaller and smaller positive numbers), we know that $\ln(x)$ approaches $-\infty$. Thus, this limit looks like $\frac{-\infty}{-\infty}$. Thus, by L'Hôpital's Rule we get

$$\lim_{x\to0^+}\frac{\ln(e^x-1)}{\ln(x)} \overset{\text{L'H}}{=} \lim_{x\to0^+}\frac{\frac{1}{e^x-1}\cdot e^x}{\frac{1}{x}} = \lim_{x\to0^+}\frac{xe^x}{e^x-1}.$$

Now when we try to plug in $x=0$, we get the indeterminate form $\frac{0}{0}$. So we can use L'Hôpital's Rule again. Now we get

$$\lim_{x\to0^+}\frac{\ln(e^x-1)}{\ln(x)} \overset{\text{L'H}}{=} \lim_{x\to0^+}\frac{xe^x}{e^x-1} \overset{\text{L'H}}{=} \lim_{x\to0^+}\frac{xe^x+e^x}{e^x} = \lim_{x\to0^+}x+1=1.$$

From this example we can see that sometimes the indeterminate forms can change as we use L'Hôpital's Rule and simplify. As long as we are careful and check at each step whether we can use L'Hôpital's Rule or not, we can still get to the answer.

1.3.4 $\infty - \infty$ Form

Question: Why is $\infty - \infty$ indeterminate? In general $\infty -$ (stuff) $= \infty$, but (stuff) $- \infty = -\infty$. So who wins? Or do they balance out and we get something like $-\pi$?

- To start with, let's look at $\lim_{x\to\infty} \ln(4x^2-6)-\ln(-x+3x^2+5)$. We know that the end behavior of $\ln(x)$ approaches infinity as x gets larger and larger. Since the insides of both logs approaches infinity, the limit looks like $\infty - \infty$. To move forward with computing the limit, we can use our logarithm rules to simplify and get

$$\lim_{x\to\infty} \ln(4x^2-6)-\ln(-x+3x^2+5) = \lim_{x\to\infty} \ln\left(\frac{4x^2-6}{-x+3x^2+5}\right).$$

Now recall that $\ln(x)$ is a continuous function on its domain. That means we can pass the limit to the inside of the function, and only have to worry about what happens with the rational function on the inside. In other words,

$$\lim_{x\to\infty} \ln\left(\frac{4x^2-6}{-x+3x^2+5}\right) = \ln\left(\lim_{x\to\infty}\frac{4x^2-6}{-x+3x^2+5}\right).$$

Notice that the inside limit is something we spent a lot of time understanding how to compute at the beginning of the class! So we can use our usual limit techniques to compute this. In fact, this is also in the $\frac{\infty}{\infty}$ form, so we could even use L'Hôpital's Rule from the earlier section! Thus, we finally conclude that the answer we are looking for is

$$\lim_{x\to\infty} \ln\left(\frac{4x^2-6}{-x+3x^2+5}\right) = \ln\left(\lim_{x\to\infty}\frac{4x^2-6}{-x+3x^2+5}\right) = \ln\left(\lim_{x\to\infty}\frac{4-\frac{6}{x^2}}{\frac{-1}{x}+3+\frac{5}{x^2}}\right) = \ln\left(\frac{4}{3}\right).$$

- For a second example, we'll compute $\lim_{x\to1^+} \frac{1}{x-1}-\frac{1}{\ln x}$. To see that this really is in $\infty - \infty$ form, notice that as x approaches 1 from the right, $\ln(x)$ will approach zero from the right. Thus, the denominator of both pieces of the limit approaches zero from the right, and we know from our parent functions that $\lim_{x\to0^+}\frac{1}{x} = \infty$. Therefore we really are in the $\infty - \infty$ form. This time only one side has a logarithm, so we can't use our log rules right off the bat. However, this really just looks like a fraction subtracted from another fraction, and we know how to simplify that with a common denominator. So we get

$$\lim_{x\to1^+} \frac{1}{x-1}-\frac{1}{\ln x} = \lim_{x\to1^+}\frac{\ln(x)-(x-1)}{(x-1)\ln(x)}.$$

When we look at this limit, we see that we are now in $\frac{0}{0}$ form, which is perfect for L'Hôpital's! Notice that on the bottom there are two functions of x that are being multiplied, so when we do the derivative we will need the product rule. Thus,

$$\begin{aligned}
\lim_{x\to1^+} \frac{1}{x-1}-\frac{1}{\ln x} &= \lim_{x\to1^+}\frac{\ln(x)-(x-1)}{(x-1)\ln(x)} \\
&\overset{\text{L'H}}{=} \lim_{x\to1^+}\frac{\frac{1}{x}-1}{\ln(x)+(x-1)\frac{1}{x}} \\
&\overset{\text{L'H}}{=} \lim_{x\to1^+}\frac{\frac{-1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{-1}{1+1} = \frac{-1}{2}.
\end{aligned}$$

1.3.5 ∞^0 Form

Question: Why is ∞^0 indeterminate? In general ∞ raised to any positive power should be equal to ∞, ∞ raised to a negative power is 0, and anything raised to the zero should be equal to 1. So who wins?

- An example of this form is the limit $\lim_{x\to\infty}(\ln(x))^{1/x}$. Notice that as x gets large, $\ln(x)$ also gets large, and that $\frac{1}{x}$ gets small. So the power converges to 0 but the function on the inside diverges to ∞. The issue with this form is that we can't do much with manipulating the exponent. So we would really like to get the $\frac{1}{x}$ out of the exponent so that we can deal with it more effectively. We will exploit our log rules here to bring that power down. However, we have to be careful! We can't just use the log rule $\ln(a^b) = b\ln(a)$ right now because it's not in that form! First, we need to transform it into that form. As with all our other techniques, we need ot change the equation without changing the outcome. Before we have been multiplying by weird forms of 1, or we have added weird forms of zero. This time, we will use the fact that e^x and $\ln(x)$ are inverses of each other. Recall that $e^{\ln(x)} = x$. That is, if we apply both $\ln(x)$ and e^x to a function, we end up with exactly the same thing. For us, that means we will do the following

$$\lim_{x\to\infty}(\ln(x))^{1/x} = \lim_{x\to\infty} e^{\ln\left((\ln(x))^{1/x}\right)}.$$

Now we can use our log rule to bring the power down, but notice that everything will be happening in the exponent of e. Since e^x is a continuous function, we can also push the limit up into the exponent. Thus, we have

$$\lim_{x\to\infty} e^{\ln\left((\ln(x))^{1/x}\right)} = e^{\lim_{x\to\infty}\frac{1}{x}\cdot\ln(\ln(x))} = e^{\lim_{x\to\infty}\frac{\ln(\ln(x))}{x}}.$$

Now we only need to worry about the limit that is in the exponent. In this case, as x goes to ∞, both the top and bottom go to ∞, so we are in the proper form for L'Hôpital's Rule. Thus, we have

$$\begin{aligned}\lim_{x\to\infty} e^{\ln\left((\ln(x))^{1/x}\right)} &= e^{\lim_{x\to\infty}\frac{\ln(\ln(x))}{x}}\\ &\overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{1}{\ln(x)}\cdot\frac{1}{x}}\\ &= e^0 = 1.\end{aligned}$$

- Now let's compute $\lim_{\to x\to\infty} x^{1/\ln(x)}$. Recall that as x tends to infinity, so does $\ln(x)$. So $\frac{1}{\ln(x)}$ approaches zero as x goes to infinity. Thus, we are in the form ∞^0. Again we need to be able to get into that exponent, so we use the same trick with e^x and $\ln(x)$. Then we get

$$\begin{aligned}\lim_{x\to\infty} x^{1/\ln(x)} &= \lim_{x\to\infty} e^{\ln(x^{1/\ln(x)})}\\ &= e^{\lim_{x\to\infty}\ln(x^{1/\ln(x)})}\\ &= e^{\lim_{x\to\infty}\frac{1}{\ln(x)}\ln(x)}\\ &= e^{\lim_{x\to\infty}1} = e.\end{aligned}$$

It turns out we don't even technically need L'Hôpital's Rule here because the logarithms cancel before we even need to take derivatives. However, we can use it twice to get the same value as the cancellation.

1.3.6 1^∞ Form

Question: Why is 1^∞ indeterminate? Usually 1 raised to any power is just equal to 1. But fractions raised to the ∞ goes to zero, and numbers larger than 1 raised to the ∞ should go off to ∞. So where does 1^∞ go?

- Consider the limit $\lim_{x\to 0^+}(e^x + x)^{1/x}$. Since $e^0 = 1$, we can see that this limit is of the form we want, 1^∞. We can see here that we are in the same situation as last time: we have an exponent that has an x in it and we need to move it around to be able to deal with it. We can try the same technique as last time and exploit the inverse property of e^x and $\ln(x)$, namely that $e^{\ln(x)} = x$. Thus, we get

$$\lim_{x\to 0^+}(e^x + x)^{1/x} = \lim_{x\to 0^+} e^{\ln\left((e^x+x)^{1/x}\right)} = \lim_{x\to 0^+} e^{\frac{\ln(e^x+x)}{x}}.$$

Here notice that as x approaches zero, the top of the fraction in the exponent approaches $\ln(1) = 0$, and the denominator approaches zero as well. Thus, we are in the earlier case of $\frac{0}{0}$. Therefore, we can move forward and use L'Hôpital's Rule to get

$$\begin{aligned}\lim_{x\to 0^+}(e^x + x)^{1/x} &= \lim_{x\to 0^+} e^{\frac{\ln(e^x+x)}{x}} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0^+} e^{\frac{1}{e^x+x}\cdot(e^x+1)} \\ &= e^{\lim_{x\to 0^+}\frac{e^x+1}{e^x+x}} = e^2.\end{aligned}$$

- For a second example, let's compute $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$. Here we can easily see that the exponent goes to infinity as x grows large, and notice that $\frac{x+2}{x-1}$ converges to 1 by L'Hôpital's Rule. Thus we are in the form 1^∞. As before, we use our logarithm trick to get

$$\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x = \lim_{x\to\infty} e^{x\ln\left(\frac{x+2}{x-1}\right)}.$$

Notice that we now have the form $\infty \cdot 0$ in the exponent, se we can use our earlier tricks to solve this exponent. Thus,

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x &= \lim_{x\to\infty} e^{x\ln\left(\frac{x+2}{x-1}\right)} \\ &= \lim_{x\to\infty} e^{\frac{\ln\left(\frac{x+2}{x-1}\right)}{1/x}} \\ &\overset{\text{L'H}}{=} \lim_{x\to\infty} e^{\frac{\frac{x-1}{x+2}\cdot\frac{(x-1)-(x+2)}{(x-1)^2}}{-1/x^2}} \\ &= \lim_{x\to\infty} e^{\frac{3x^2}{(x+2)(x-1)}} = e^3.\end{aligned}$$

1.3.7 0^0 Form

Question: Why is 0^0 indeterminate? In general zero raised to any positive power is just zero, but but anything raised to the zero should be equal to 1. So which is it?

- Consider $\lim_{x\to 0^+} x^{\frac{-1}{\ln(x)}}$. To check that this is in the right form, we need ot look at the exponent. as x approaches 0, $\ln(x)$ approaches negative infinity. Then 1 divided by something that approaches approaches infinity goes to zero. So we are in 0^0 form. We again have functions in the exponent, so we'll use the logarithm trick. Thus,

$$\begin{aligned}\lim_{x\to 0^+} x^{\frac{-1}{\ln(x)}} &= \lim_{x\to 0^+} e^{\ln\left(x^{-1/\ln(x)}\right)}\\ &= \lim_{x\to 0^+} e^{\frac{\ln(x)}{-\ln(x)}}\\ &\overset{\text{L'H}}{=} \lim_{x\to 0^+} e^{\frac{1/x}{-1/x}}\\ &\overset{\text{L'H}}{=} e^{1/-1} = e^{-1}.\end{aligned}$$

  We could have just simplified $\frac{\ln(x)}{-\ln(x)}$ to be -1 right off the bat, but we used L'Hôpital's Rule to show that it still works.

- For another example, we'll compute $\lim_{x\to 0^+} x^{x^{10}}$. Since we don't have a ratio, we use are usual transformation to get one to show up. To start,

$$\lim_{x\to 0^+} x^{x^{10}} = \lim_{x\to 0^+} e^{\ln(x^{x^{10}})} = \lim_{x\to 0^+} e^{x^{10}\ln(x)}.$$

  We now have a $0\cdot\infty$ form, which we dealt with above. To continue, we get

$$\begin{aligned}\lim_{x\to 0^+} x^{x^{10}} &= \lim_{x\to 0^+} e^{x^{10}\ln(x)}\\ &= \lim_{x\to 0^+} e^{\frac{\ln(x)}{x^{-10}}}\\ &\overset{\text{L'H}}{=} \lim_{x\to 0^+} e^{\frac{1/x}{-10x^{-11}}}\\ &= \lim_{x\to 0^+} e^{\frac{-x^{10}}{10}} = 1.\end{aligned}$$

- As our last example, consider $\lim_{x\to 0^+} x^{\sin(x)}$. Since $\sin(0) = 0$, we are in the form 0^0. As before, we get

$$\lim_{x\to 0^+} x^{\sin(x)} = \lim_{x\to 0^+} e^{\sin(x)\ln(x)} = \lim_{x\to 0^+} e^{\frac{\ln(x)}{\csc(x)}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} e^{\frac{1/x}{-\csc(x)\cot(x)}}.$$

This is in the form $\frac{\infty}{\infty}$, but before we blindly keep using L'Hôpital's rule, we should simplify the expression. Thus, we get

$$\lim_{x\to 0^+} e^{\frac{1/x}{-\csc(x)\cot(x)}} = \lim_{x\to 0^+} e^{\frac{-\sin(x)\sin(x)}{x\cos(x)}},$$

which is in $\frac{0}{0}$ form. However, we should recognize the limit $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. When we incorporate this into the calculation, we don't even need L'Hôpital's Rule any more (but we could if desired). Thus, we get our final answer of

$$\lim_{x\to 0^+} e^{\frac{-\sin(x)\sin(x)}{x\cos(x)}} = e^{-1\cdot\frac{0}{1}} = 1.$$

Unit-1

CHAPTER 1

SUCCESSIVE DIFFERENTIATION AND LEIBNITZ'S THEOREM

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), \ldots, f^{(n)}(x)$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

⋮

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

1.2 Calculation of n^{th} Derivatives

i. **n^{th} Derivative of e^{ax}**

Let $y = e^{ax}$

$y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

⋮

$y_n = a^n e^{ax}$

ii. **n^{th} Derivative of $(ax+b)^m$, m is a +ve integer greater than n**

Let $y = (ax+b)^m$

$y_1 = m\,a(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

⋮

$y_n = m(m-1)\ldots(m-n+1)a^n(ax+b)^{m-n}$

$$= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$$

iii. **n^{th} Derivative of $y = \log(ax+b)$**

Let $y = \log(ax+b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\,a^3}{(ax+b)^3}$$

$\vdots$

$$y_n = (-1)^{n-1} \frac{(n-1)!\,a^n}{(ax+b)^n}$$

iv. **n^{th} Derivative of $y = \sin(ax+b)$**

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$\vdots$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax+b)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

v. **n^{th} Derivative of $y = e^{ax}\sin(ax+b)$**

Let $y = e^{ax}\sin(bx+c)$

$$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)]$$

Putting $a = r\cos\alpha,\ b = r\sin\alpha$

$$= e^{ax}\ r\ \sin(bx+c+\alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx+c+2\alpha)$

$\vdots$

$$y_n = e^{ax}\ r^n\ \sin(bx+c+n\alpha)$$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$

$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & m > 0, m > n \\ 0, \quad m > 0, & m < n, \\ n!\, a^n, & m = n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, & m = -1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$
$y = e^{ax}\sin(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$
$y = e^{ax}\cos(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow y_n = (-1)^{n+1} n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let y $= \sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10\,x + \cos 2\,x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of $sin^2x cos^3x$

Solution: Let y $= sin^2x cos^3x$

$= sin^2x cos^2x\ cosx$

$= \frac{1}{4} sin^2 2x\ cosx = \frac{1}{8}(1 - cos4x)cosx$

$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$

$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$

$= \frac{1}{16}(2cosx - cos3x - cos5x)$

$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$

$= \left(\frac{1}{2} 2\ sin^2x\right)^2$

$= \frac{1}{4}((1 - \cos 2x)^2$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$

$= \frac{3}{8} - \frac{1}{2}cos2x + \frac{1}{8}cos4x$

$\therefore y_n = -\frac{1}{2}2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n cos\left(4x + \frac{n\pi}{2}\right)$

Example 5 Find the n^{th} derivative of $e^{3x} cosx\ \sin^2 2x$

Solution: Let $y = e^{3x} cosx\ \sin^2 2x$

Now $\cos x \sin^2 2x = \frac{1}{2}(\cos x - \cos x \cos 4x)$

$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x + \cos 3x\right)$

$\Rightarrow y = e^{3x} cosx\ \sin^2 2x = \frac{1}{2}e^{3x}\cos x - \frac{1}{4}e^{3x}\cos 5x - \frac{1}{4}e^{3x}\cos 3x$

$\therefore y_n = \frac{1}{2}e^{3x}(9+1)^{\frac{n}{2}} \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}(9+25)^{\frac{n}{2}} \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$

$- \frac{1}{4}e^{3x}(9+9)^{\frac{n}{2}} \cos\left(3x + n\tan^{-1}\frac{3}{3}\right)$

$= \frac{1}{2}e^{3x}\, 10^{\frac{n}{2}} \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}\, 34^{\frac{n}{2}} \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$

$- \frac{1}{4}e^{3x}\, 18^{\frac{n}{2}} \cos(3x + n\tan^{-1} 1)$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$\therefore y_n = a^n\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$

$$= a^n\left[\left\{\sin\left(ax+\frac{n\pi}{2}\right)+\cos\left(ax+\frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= a^n\left[\sin^2\left(ax+\frac{n\pi}{2}\right)+\cos^2\left(ax+\frac{n\pi}{2}\right)+2\sin\left(ax+\frac{n\pi}{2}\right).\cos\left(ax+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= a^n[1+\sin(2ax+n\pi)]^{\frac{1}{2}}$$

$$= a^n[1+\sin 2ax\cos n\pi+\cos 2ax\sin n\pi\,]^{\frac{1}{2}}$$

$$= a^n[1+(-1)^n\sin 2ax\,]^{\frac{1}{2}} \quad \because \cos n\pi=(-1)^n \text{ and } \sin n\pi=0$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y=\tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1=\frac{dy}{dx}=\frac{1}{a\left(1+\frac{x^2}{a^2}\right)}=\frac{a}{x^2+a^2}=\frac{a}{x^2-(ai)^2}$$

$$=\frac{a}{(x+ai)(x-ai)}=\frac{a}{2ai}\left(\frac{1}{x-ai}-\frac{1}{x+ai}\right)$$

$$=\frac{1}{2i}\left(\frac{1}{x-ai}-\frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n=\frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}-\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x=r\cos\theta,\ a=r\sin\theta$ such that $\theta=\tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n=\frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta-i\sin\theta)^n}-\frac{1}{r^n(\cos\theta+i\sin\theta)^n}\right]$$

$$=\frac{(-1)^{n-1}(n-1)!}{2ir^n}[(\cos\theta-i\sin\theta)^{-n}-(\cos\theta+i\sin\theta)^{-n}]$$

Using De Moivre's theorem, we get

$$y_n=\frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta]$$

$$=\frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$=\frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a=r\sin\theta$$

$$=\frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\,\sin^n\theta \qquad \text{where } \theta=\tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y=\frac{1}{1+x+x^2}$

$$=\frac{1}{(x-w)(x-w^2)} \quad \text{where } w=\frac{-1+i\sqrt{3}}{2} \text{ and } w^2=\frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y=\frac{1}{w-w^2}\left(\frac{1}{x-w}-\frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{(2x+1-i\sqrt{3})^{n+1}} - \frac{1}{(2x+1+i\sqrt{3})^{n+1}}\right]$$

Substituting $2x+1 = r\,cos\theta$, $\sqrt{3} = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}(-1)^{n+1} n!}{\sqrt{3}\;r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}(-1)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\left[cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta\right]$$

$$\because \sqrt{3} = r\,sin\theta$$

$$= \frac{i\,2^{n+1}(-1)^{n+1} n!}{(\sqrt{3})^{n+2}}\; 2i\,sin(n+1)\theta\, sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\; sin(n+1)\theta\, sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x\,(\sec x \tan x) = 2\,sec^2 x \tan x$$

$$\therefore \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\,sec^2 x \tan x - 2(x + \tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow \left(\sqrt{1+x^2}\right)\frac{dy}{dx} = 1$

Differentiating both sides w.r.t. x , we get

$\left(\sqrt{1+x^2}\right)\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = 0$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Exercise 1 A

1. Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$

   Ans. $(-1)^n n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$

2. Find the n^{th} derivative of $\cos x \cos 2x \cos 3x$

   Ans. $\frac{1}{4}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 6^n \cos\left(6x + \frac{n\pi}{2}\right)\right]$

3. If $x = \sin t$, $y = \sin at$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$

4. If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, show that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

5. If $y = \frac{x}{x^2+a^2}$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{(-1)^n n!}{a^{n+1}} \cos(n+1)\theta \sin^{n+1}\theta$ where $\theta = \tan^{-1}\frac{x}{a}$

6. If $y = e^x \sin^2 x$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{1}{2} e^x\left[1 - 16\left(2x + \frac{n\pi}{2}\right)\right]$

7. Find n^{th} differential coefficient of $y = \log[(ax+b)(cx+d)]$

   Ans. $y_n = (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right]$

8. If $y = x\log\frac{x-1}{x+1}$, show that $y_n = (-1)^{n-1}(n-2)!\left[\frac{x-n}{(x-1)^n} + \frac{x+n}{(x+1)^n}\right]$

9. If $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, show that $y_n = \frac{1}{2}(-1)^{n-1}(n-1)! \sin n\theta \sin^n\theta$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1}\, u_{n-1}\, v_1 + n_{C_2} u_{n-2}\, v_2 + \cdots + n_{C_r}\, u_{n-r}\, v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1} \frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1}\, u_{n-1}\, v_1 + n_{C_2} u_{n-2}\, v_2 + \cdots + n_{C_r}\, u_{n-r}\, v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n]$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$

$$= e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1}\, u_{n-1}\, v_1 + n_{C_2} u_{n-2}\, v_2 + \cdots + n_{C_r}\, u_{n-r}\, v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x} \sin 4x\right)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) +$$

$$2nxe^{3x}\, 5^{n-1} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) +$$

$$n(n-1) e^{3x}\, 5^{n-2} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right) + 0$$

$$= e^{3x}5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13 If $y = a\cos(\log x) + b\,sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\,cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\,sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{acos(logx) + b\,sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$[y_{n+2}(1+x^2) + {}^nc_1 y_{n+1}2x + {}^nc_2 y_n.2] + (y_{n+1}x + {}^nc_1 y_n.1) = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Example 15 If $y = \sin(m\, sin^{-1}x)$, show that $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y = \sin(m\, sin^{-1}x)$ ......①

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\, sin^{-1}x)$ ......②

$\Rightarrow (1-x^2){y_1}^2 = m^2\cos^2(m\, sin^{-1}x)$

$\Rightarrow (1-x^2){y_1}^2 = m^2[1-\sin^2(m\, sin^{-1}x)]$

$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)$......③

$\Rightarrow (1-x^2){y_1}^2 + m^2y^2 = m^2$

Differentiating w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^2 2yy_1 = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$

Using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + {}^nc_1 y_{n+1}(-2x) + {}^nc_2 y_n(-2)] - (y_{n+1}x + {}^nc_1 y_n 1) + m^2y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$......④

Putting $x = 0$ in ①, ②and ③

$y(0) = 0, y_1(0) = m$ and $y_2(0) = 0$

Putting $x = 0$ in ④

$y_{n+2}(0) = (n^2-m^2)y_n(0)$

Putting $n = 1,2,3$ ... ... in the above equation, we get

$y_3(0) = (1^2-m^2)y_1(0)$

$=(1^2-m^2)m \quad \because y_1(0) = m$

Scanned with OKEN Scanner

$$y_3(0) = (1^2 + m^2)y_1(0)$$

$$=(1^2 + m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0)$$

$$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$$

$$y_5(0) = (3^2 + m^2)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ odd \end{cases}$$

Example 17 If $y = tan^{-1}x$, show that

$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x$.....①

$$\Rightarrow y_1 = \frac{1}{1+x^2} \quad \ldots\ldots ②$$

$$y_2 = \frac{-2x}{1+x^2}$$

$$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0 \ldots\ldots ③$$

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1 + x^2) + n_{C_1}y_{n+1}(2x) + n_{C_2}y_n(2)] + 2(y_{n+1}x + n_{C_1}y_n 1) = 0$$

$$\Rightarrow y_{n+2}(1 + x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0, y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = -n(n+1)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = -1(2)y_1(0)$

$= -2 \qquad \because y_1(0) = 1$

$y_4(0) = -2(3)y_2(0)$

$= 0 \qquad \because y_2(0) = 0$

$y_5(0) = -3(4)y_3(0)$

$= -3(4)(-2) = 4!$

$y_6(0) = -4(5)y_4(0) = 0$

$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$

$\vdots$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $\quad y_{2n}(0) = 0$

Example18 If $y = (sin^{-1}x)^2$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$(1 - x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1 - x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1 - x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1 - x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1 - x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n 1) = 0$

$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2.2^2.4^2 \ldots\ldots\ldots (n-2)^2, & \text{if } n \text{ is even} \end{cases}$$

Exercise 1 B

1. Find y_n, if $y = x^3 \cos x$

Ans. $x^3 \cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 \cos\left[x + \frac{1}{2}(n-1)\pi\right] + 3n(n-1)x \cos\left[x + \frac{1}{2}(n-2)\pi\right] + n(n-1)(n-2)\cos\left[x + \frac{1}{2}(n-3)\pi\right]$

2. Find y_n, if $y = x^2 e^x \cos x$

Ans. $2^{\frac{n}{2}} e^x \cos\left(x + \frac{n\pi}{4}\right)$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right)$, prove that

$$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$$

5. If $y = \left[x + \sqrt{1+x^2}\right]^m$, prove that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

6 If $y = (\sinh^{-1} x)^2$, show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $y_n(0)$.

Ans. $y_{2n+1}(0) = 0$ and $y_{2n}(0) = (-1)^{n-1} 2.2^2.4^2 \ldots\ldots\ldots (2n-2)^2$

7. If $y = \cos(m \sin^{-1} x)$, show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$. Also find $y_n(0)$.

8. If $f(x) = \tan x$, prove that $f^n(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) - \cdots = \sin\frac{n\pi}{2}$

Taylor's Theorem: Introductory Waffle

Suppose that we once went on a journey, starting at time a at position $f(a)$ and finishing at time x at position $f(x)$. Later, we decide to recreate this journey, but we've forgotten exactly what we did last time and so we try to approximate it.

For our first attempt, we will travel at constant speed k for the whole journey. We travel at speed k from time a to time t, and so we travel a distance of $(t-a)k$, and arrive at position

$$f(a) + (t-a)k$$

Our aim is to arrive at position $f(x)$ at time x, so we need to choose k so that $f(x) = f(a) + (x-a)k$. It's clear from this that the correct speed k to use is $(f(x) - f(a))/(x-a)$. We might then recall the Mean Value Theorem, which says that this fraction equals $f'(c)$ for some $c \in (a, x)$. In other words, the constant speed k we need to use happens to be an actual speed we had at some point during our original journey.

Let's see another way that this result could have been reached. Since we're still leaving at the same time, and we want to arrive at the same time, let's define a function to measure the difference between our positions at time t in both the original and current journeys.

Let $\phi(t) = f(t) - f(a) - (t-a)k$.

Then $\phi(a) = 0$, and our desired value of k is such that $\phi(x) = 0$ as well. By Rolle's Theorem there is some $c \in (a, x)$ such that $\phi'(c) = 0$. Since $\phi'(t) = f'(t) - k$, we have $f'(c) = k$. Hence

$$f(x) = f(a) + (x-a)f'(c) \qquad \text{for some } c \in (a, x)$$

And we have obtained the same result.

We now decide that constant speed isn't a very accurate representation of our original journey. So we'll try to mimic it more closely this time. We'll start at time a at position $f(a)$ as before, only this time we'll start off travelling at our original initial speed $f'(a)$. We'll now try to apply a constant acceleration k to adjust our speed so that we arrive at $f(x)$ at time x. Thinking of $s = ut + \frac{1}{2}at^2$, we find that from time a to time t we travel a distance of $(t-a)f'(a) + \frac{1}{2}(t-a)^2k$, and so at time t we arrive at position

$$f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2}k$$

Our aim is to arrive at $f(x)$ at time x. So, as before, we'll define a difference function, measuring the difference between our positions at time t in both the original and current journeys.

Let $\phi(t) = f(t) - f(a) - (t-a)f'(a) - \dfrac{(t-a)^2}{2}k$.

Then $\phi(a) = 0$, and our desired value of k is such that $\phi(x) = 0$ as well. By Rolle's Theorem, there is some $c_1 \in (a, x)$ such that $\phi'(c_1) = 0$.

Now consider $\phi'(t) = f'(t) - f'(a) - (t-a)k$. Since $\phi'(a) = \phi'(c_1) = 0$, by a second application of Rolle there is some $c \in (a, c_1)$ such that $\phi''(c) = 0$. Since $\phi''(t) = f''(t) - k$, we see that $k = f''(c)$. Thus we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(c) \qquad \text{for some } c \in (a, x)$$

Next, we decide to make our journey even more faithful to the original: we start at $f(a)$ at time a, travelling at the original speed $f'(a)$ and with original acceleration $f''(a)$, and we try to adjust the whatever-the-derivative-of-acceleration-is so that we arrive at $f(x)$ at time x.

And so on.

Scanned with OKEN Scanner

Taylor's Theorem with the Cauchy Remainder

Often when using the Lagrange Remainder, we'll have a bound on $f^{(n)}$, and rely on the $n!$ beating the $(x-a)^n$ as $n \to \infty$. But if the $f^{(n)}$ term starts providing us with an $n!$-shaped term on top, such as with a binomial expansion, then we might need a better expression than $(x-a)^n$.

We'll show that $R_n(x) = \dfrac{(x-c)^{n-1}(x-a)}{(n-1)!} f^{(n)}(c)$, for some $c \in (a, x)$. (A different c, of course.)

This will allow us better estimates if all we have is a general bound on $f^{(n)}$, since $|x-c| < |x-a|$.

We have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n(x)$$

Rearranging

$$R_n(x) = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2} f''(a) - \ldots - \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a).$$

We want to estimate this, so let's turn it into a function of some variable t, by setting

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2} f''(t) - \ldots - \frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(t).$$

so that $F(a) = R_n(x)$ and $F(x) = 0$.

Then, noting that

$$\frac{d}{dt}\left(-\frac{(x-t)^m}{m!} f^{(m)}(t)\right) = \frac{(x-t)^{m-1}}{(m-1)!} f^{(m)}(t) - \frac{(x-t)^m}{m!} f^{(m+1)}(t),$$

we have, by a telescoping sum,

$$F'(t) = -\frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)$$

Applying the Mean Value Theorem to F on the interval $[a, x]$, we find some $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{x - a} = F'(c) = -\frac{(x-c)^{n-1}}{(n-1)!} f^{(n)}(c)$$

Rearranging gives

$$F(a) = F(x) + \frac{(x-c)^{n-1}(x-a)}{(n-1)!} f^{(n)}(c)$$

That is, as claimed,

$$R_n(x) = \frac{(x-c)^{n-1}(x-a)}{(n-1)!} f^{(n)}(c)$$

This result is **Taylor's Theorem with the Cauchy remainder.**

Taylor's Theorem with the Lagrange Remainder

Continuing the thoughts from our introductory waffle, we want k such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k.$$

Let $\phi(t) = f(t) - f(a) - (t-a)f'(a) - \frac{(t-a)^2}{2}f''(a) - \ldots - \frac{(t-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{(t-a)^n}{n!}k.$

Our k is such that $\phi(a) = \phi(x) = 0$, and so Rolle's Theorem gives us $c_1 \in (a, x)$ such that $\phi'(c_1) = 0$.

Now, $\phi'(t) = f'(t) - f'(a) - (t-a)f''(a) - \ldots - \frac{(t-a)^{n-2}}{(n-2)!}f^{(n-1)}(a) - \frac{(t-a)^{n-1}}{(n-1)!}k.$

Since $\phi'(a) = \phi'(c_1) = 0$, a second application of Rolle's Theorem gives us $c_2 \in (a, c_1)$ such that $\phi''(c_2) = 0$.

Now, $\phi''(t) = f''(t) - f''(a) - (t-a)f'''(a) - \ldots - \frac{(t-a)^{n-3}}{(n-3)!}f^{(n-1)}(a) - \frac{(t-a)^{n-2}}{(n-2)!}k.$

Since $\phi''(a) = \phi''(c_2) = 0$, a third application of Rolle's Theorem gives us $c_3 \in (a, c_2)$ such that $\phi''(c_3) = 0$.

And so on. We reach $\phi^{(n)}(t) = f^{(n)}(t) - k$, and find some $c \in (a, c_{n-1}) \subset (a, x)$ such that $\phi^{(n)}(c) = 0$. I.e., such that $k = f^{(n)}(c)$.

Since $\phi(x) = 0$ for this k, we conclude that, for some $c \in (a, x)$, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(c).$$

We might prefer to make the following change of notation. Let $x = a + h$, and since $c \in (a, x)$, write $c = a + \theta h$ where $\theta \in (0, 1)$. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a + \theta h).$$

Examining what we did, we see that f needed to be n-times differentiable on (a, x), but at a itself we needed f to be only $(n-1)$-times differentiable, although with $f^{(n-1)}$ continuous there. And we didn't need $f^{(n)}$ to be continuous anywhere.

This result is **Taylor's Theorem with the Lagrange remainder.**

The remainder is $R_n(x) = \frac{(x-a)^n}{n!}f^{(n)}(c).$

Taylor's Theorem with the Integral Remainder

There is another form of the remainder which is also useful, under the slightly stronger assumption that $f^{(n)}$ is continuous.

We'll show that $R_n = \displaystyle\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$.

The proof of this is by induction, with the base case being the Fundamental Theorem of Calculus.

When $n = 1$, we have $f(x) = f(a) + \displaystyle\int_a^x f'(t)\,dt$, so we're done by the FTC.

Assuming the formula above for the $(n-1)$th case, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-2}}{(n-2)!}f^{(n-2)}(a) + R_{n-1},$$

where

$$\begin{aligned} R_{n-1} &= \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} f^{(n-1)}(t)\,dt \\ &= \left[-\frac{(x-t)^{n-1}}{(n-1)(n-2)!} f^{(n-1)}(t)\right]_a^x + \int_a^x \frac{(x-t)^{n-1}}{(n-1)(n-2)!} f^{(n)}(t)\,dt \\ &= \frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt \end{aligned}$$

and substituting this into the expression for $f(x)$ gives the required form for $R_n(x)$.

This result is **Taylor's Theorem with the integral form of the remainder.**

We can derive the Lagrange remainder from this. Since $f^{(n)}$ is continuous on $[a, x]$, it is bounded there and attains its bounds. Say $m \leqslant f^{(n)}(t) \leqslant M$, with these bounds being attained.

$$m\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,dt \;\leqslant\; \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt \;\leqslant\; M\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

Since

$$\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,dt = \frac{(x-a)^n}{n!},$$

we have

$$m\frac{(x-a)^n}{n!} \;\leqslant\; R_n(x) \;\leqslant\; M\frac{(x-a)^n}{n!}$$

Applying the Intermediate Value Theorem, we find some $c \in (a, x)$ such that

$$R_n(x) = \frac{(x-a)^n}{n!} f^{(n)}(c)\,.$$

11.10 Taylor and Maclaurin Series

The idea is to obtain a good approximation to a function $f(x)$ among all polynomials of degree n. There are many sensible notions of what 'good approximation' could mean. The notion here is that we want our approximating polynomial to share the value and first n derivatives with $f(x)$ at a point $x = a$.

Definition. Suppose that f is a function which is n-times differentiable at $x = a$. The nth Taylor polynomial of f centered at $x = a$ is the polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Observe the following:

- T_n is a polynomial of degree $\leq n$ (it is possible to have $f^{(n)}(a) = 0$ so that T_n has degree $< n$).
- $T_0(x) = f(a)$
- $T_1(x) = f(a) + f'(a)(x-a)$ is the linear approximation/tangent line to $y = f(x)$ at $x = a$. The Taylor polynomials are, essentially, higher order versions of the linear approximation.

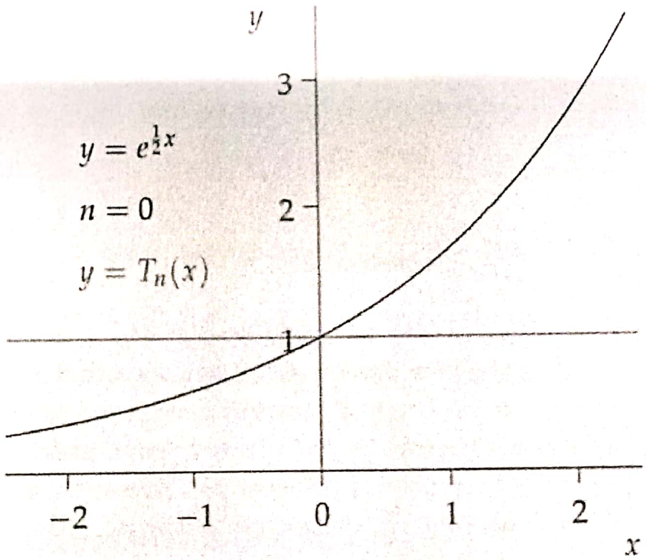
Example Let $f(x) = e^{\frac{1}{2}x}$. Then $f^{(k)}(x) = \left(\frac{1}{2}\right)^k e^{\frac{1}{2}x}$, so $f^{(k)}(0) = \left(\frac{1}{2}\right)^k$. The first few Taylor polynomials of f centered at zero are therefore

$$T_0(x) = 1, \qquad T_1(x) = 1 + \frac{1}{2}x, \qquad T_2(x) = 1 + \frac{1}{2}x + \frac{1}{2^2 \cdot 2!}x^2 = 1 + \frac{1}{2}x + \frac{1}{8}x^2$$
$$T_3(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{2^3 \cdot 3!}x^3 = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3$$

More generally,

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{2^k k!}x^k$$

The animation shows the Taylor polynomials of $f(x) = e^{\frac{1}{2}x}$ centered at $x = 0$ for $n = 0, 1, 2, 3$ and 4.



Example Let us repeat the example with $f(x) = \sin x$, first centered at $x = 0$ and then at $x = \frac{\pi}{2}$. First we compute a few derivatives and spot a pattern:

$$f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f'''(x) = -\cos x, \qquad f^{(4)}(x) = \sin x, \dots$$

With a little thinking, it should be clear that we have

$$\begin{cases} f^{(2n)}(x) = (-1)^n \sin x \\ f^{(2n+1)}(x) = (-1)^n \cos x \end{cases} \implies \begin{cases} f^{(2n)}(0) = 0 \\ f^{(2n+1)}(0) = (-1)^n \end{cases} \quad \text{and} \quad \begin{cases} f^{(2n)}(\frac{\pi}{2}) = (-1)^n \\ f^{(2n+1)}(\frac{\pi}{2}) = 0 \end{cases}$$

The first few Taylor polynomials centered at $x = 0$ are therefore

$$T_0(x) = 0, \qquad T_1(x) = x, \qquad T_2(x) = x, \qquad T_3(x) = x - \frac{1}{3!}x^3, \qquad T_4(x) = x - \frac{1}{3!}x^3$$

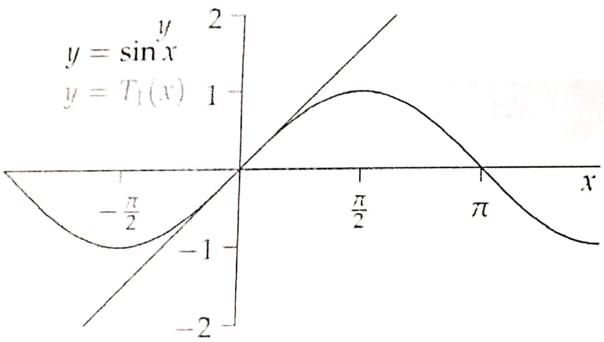
Indeed, if $2n \geq 2$ is even, then $T_{2n}(x) = T_{2n-1}(x)$ has degree $2n-1$. In general we have

$$T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots, \qquad T_{2n+2}(x) = T_{2n+1}(x)$$

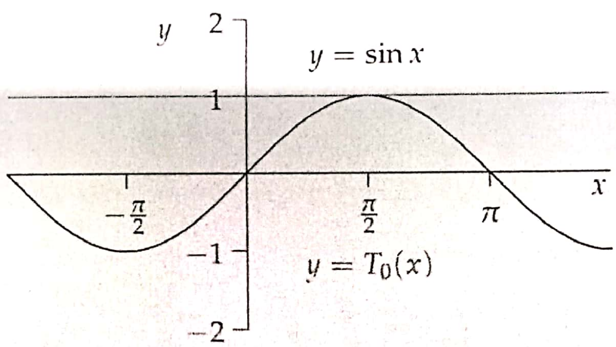
Repeating the exercise centered at $x = \frac{\pi}{2}$ we obtain

$$T_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \left(x - \tfrac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2!}\left(x - \tfrac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \tfrac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \tfrac{\pi}{2}\right)^6 + \cdots$$

The first few of these polynomials are animated below.



Centered at $x = 0$



Centered at $x = \frac{\pi}{2}$

From the pictures, it certainly seems that successive Taylor polynomials provide better approximations to the original functions, with the approximation improving the closer x is to the center a. Indeed one might say that Taylor polynomials appear to fit more snugly into the curvature of the original blue curve as n increases. This reflects the fact that the first n derivatives of $T_n(x)$ at $x = a$ match those of $f(x)$ and, as the following Theorem shows, this property completely characterises the Taylor polynomials.

Theorem (**Equality of derivatives**). *Let $T_n(x)$ be the nth Taylor polynomial of $f(x)$ centered at $x = a$. Then:*

1. *The value and first n derivatives of $T_n(x)$ equal those of $f(x)$ at $x = a$. That is, for any $m = 0, 1, 2, \ldots, n$, we have*

$$T_n^{(m)}(a) = f^{(m)}(a) \qquad \left(\text{alternately } \frac{d^m}{dx^m}\bigg|_{x=a} T_n(x) = \frac{d^m}{dx^m}\bigg|_{x=a} f(x)\right)$$

2. *$T_n(x)$ is the unique degree $\leq n$ polynomial with the above property.*

Proof. 1. Note that

$$\frac{d^m}{dx^m}(x-a)^k = \begin{cases} k(k-1)\cdots(k-m+1)(x-a)^{k-m} & \text{if } m \leq k \\ 0 & \text{if } m > k \end{cases}$$

$$\implies \frac{d^m}{dx^m}\bigg|_{x=a}(x-a)^k = \begin{cases} k! & \text{if } m = k \\ 0 & \text{otherwise} \end{cases}$$

Since the Taylor polynomial $T_n(x)$ is a finite sum, and each coefficient $\frac{f^{(k)}(a)}{k!}$ is a constant, it is now immediate that

$$\frac{d^m}{dx^m}\bigg|_{x=a} T_n(x) = \sum_{k=0}^{n} \frac{d^m}{dx^m}\bigg|_{x=a} \frac{f^{(k)}(a)}{k!}(x-a)^k = f^{(k)}(a)$$

as required.

2. Suppose that $p(x)$ is a degree $\leq n$ polynomial which shares its value and first n derivatives at $x = a$ with $f(x)$. It is a fact[1] that $p(x)$ may be written in the form

$$p(x) = \sum_{k=0}^{n} c_k(x-a)^k$$

for some constants $c_0, \ldots, c_n$. Similarly to our calculations above, it follows that

$$p^{(m)}(x) = \sum_{k=m}^{n} c_k k(k-1)\cdots(k-m+1)(x-a)^{k-m} \implies p^{(m)}(a) = c_m m!$$

If $p^{(m)}(a) = f^{(m)}(a)$ for all $m \leq n$, then $c_m = \frac{f^{(m)}(a)}{m!}$ and so $p(x) = T_n(x)$ is the nth Taylor polynomial of $f(x)$ centered at $x = a$.

■

Now that we understand Taylor polynomials, it is a small matter to consider the power series obtained by letting $n \to \infty$.

Definition. Suppose that f is infinitely differentiable at $x = a$. The *Taylor series of f centered at $x = a$* is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If the center is $a = 0$, the Taylor series is commonly referred to as the *Maclaurin Series* of f.

[1] To be absolutely convinced of this, you need some linear algebra...

Example Find the Maclaurin series of $f(x) = \sin 2x$ and compute its interval of convergence. This is very similar to the computation of the Taylor polynomials of $y = \sin x$ above. Just be careful of the factor of 2...

If $f(x) = \sin 2x$, then the derivatives of $f(x)$ follow a pattern

$$f'(x) = 2\cos 2x, \qquad f''(x) = -2^2 \sin 2x, \qquad f'''(x) = 2^3 \cos 2x, \qquad f^{(4)}(x) = -2^4 \sin 2x, \ldots$$

With a little thinking, it should be clear that we have

$$\begin{cases} f^{(2n)}(x) = (-1)^n 2^{2n} \sin 2x \\ f^{(2n+1)}(x) = (-1)^n 2^{2n+1} \cos 2x \end{cases} \implies \begin{cases} f^{(2n)}(0) = 0 \\ f^{(2n+1)}(0) = (-1)^n 2^{2n+1} \end{cases}$$

It follows that the Maclaurin series of $\sin 2x$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$$

Its radius of convergence may be computed using the Ratio Test:

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \cdot \frac{(2n+3)!}{(-1)^{n+1} 2^{2n+3}} \right| = \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{2^2} = \infty$$

It follows that the Taylor series converges for all real numbers and the interval of convergence is $(-\infty, \infty)$.

Common Maclaurin Series

All of the following may be computed and checked exactly as in the above example.

Function	Maclaurin Series	Interval of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$	$(-\infty, \infty)$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$	$(-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$	$(-\infty, \infty)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots$	$[-1, 1)$

Using these, you can easily find power series representations for similar functions. For example $f(x) = e^{2x-4}$ has the following Taylor series centered at $x = 2$:

$$\sum_{n=0}^{\infty} \frac{(2x-4)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-2)^n = 1 + 2(x-2) + \frac{2^2}{2}(x-2)^2 + \frac{2^3}{6}(x-2)^3 + \cdots$$

Similarly, $\sin(x^2)$ has the following Maclaurin series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} = x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} + \cdots$$

Both of these examples converge on the entire real line.
Moreover, it is a Theorem that if a function equals a power series, then that series is the Taylor series for said function. Looking back at the previous section, we see, for example, that

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

where the series has interval of convergence $(-1, 1]$. This must be the Maclaurin series of $\tan^{-1} x$.

More advanced ideas

There are many outstanding questions regarding Taylor polynomials and series, some of which will be addressed in later courses. For example:

- Does a function *equal* its Taylor series on the interval of convergence?
- How good an approximation does a Taylor polynomial provide?

The answer to the first question depends on the function. For all of our common examples, the answer is yes, the argument requiring nothing more than a little differential equations. However there are plenty functions which do not equal their Taylor series. For example, a little playing with l'Hôpital's rule should convince you that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has *every derivative* $f^{(n)}(0) = 0$, whence its Maclaurin series is simply $T(x) = 0$. It follows that f only equals it Maclaurin series at $x = 0$.

An Engineer might use a Taylor polynomial approach to approximately solve a particular equation. In this situation, the second question is critical. If they know, for example, that a degree 7 Taylor polynomial will approximate the exact (yet unknown) $f(x)$ correct to 3 decimal places, perhaps this margin of error is small enough to safely design a structure. Without information like this, anything depending on the approximate 'solution' could fail. Thankfully there are several methods for estimating the error in a Taylor approximation.

Suggested problems

1. Use the standard table to find the Maclaurin series for the following functions.[2]

   (a) $f(x) = e^{3x}$

   (b) $g(x) = \cos(2x^2)$

[2] You must be able to do this *without* looking at the table — it will not be given in the exam.

(c) $h(x) = \frac{1-\cos x}{x^2}$

2. Find the Maclaurin series of the given functions *directly from the definitions* (don't just quote the standard table and manipulate!).

   (a) $f(x) = \sin(3x)$

   (b) (Harder) $g(x) = (1+x)^{1/2}$ $\left(\textit{you may find } 1\cdot3\cdot5\cdots(2n-1) = \frac{(2n-1)!}{2\cdot4\cdots(2n-2)} = \frac{(2n-1)!}{2^{n-1}(n-1)!} \textit{ useful}\right)$.

3. (a) Use the definition to find the Taylor series for e^x centered at $x = 1$.

   (b) Use the definition to find the Taylor series for $\sin x$ centered at $x = \frac{\pi}{2}$.

   (c) How could you have used the standard table of Maclaurin series to answer parts (a,b) more quickly?

   (d) Use your observation to find the Taylor series for $\sin x$ centered at $x = \frac{\pi}{6}$, *without* finding any derivatives!

Unit-2

Tangents and normals

mc-TY-tannorm-2009-1

This unit explains how differentiation can be used to calculate the equations of the tangent and normal to a curve. The tangent is a straight line which just touches the curve at a given point. The normal is a straight line which is perpendicular to the tangent.

To calculate the equations of these lines we shall make use of the fact that the equation of a straight line passing through the point with coordinates (x_1, y_1) and having gradient m is given by

$$\frac{y - y_1}{x - x_1} = m$$

We also make use of the fact that if two lines with gradients m_1 and m_2 respectively are perpendicular, then $m_1 m_2 = -1$.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- calculate the equation of the tangent to a curve at a given point
- calculate the equation of the normal to a curve at a given point

Contents

1. Introduction

Consider a function $f(x)$ such as that shown in Figure 1. When we calculate the derivative, f', of the function at a point $x = a$ say, we are finding the gradient of the tangent to the graph of that function at that point. Figure 1 shows the tangent drawn at $x = a$. The gradient of this tangent is $f'(a)$.

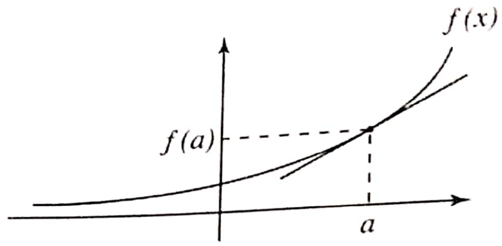


Figure 1. The tangent drawn at $x = a$ has gradient $f'(a)$.

We will use this information to calculate the equation of the tangent to a curve at a particular point, and then the equation of the normal to a curve at a point.

Key Point

$f'(a)$ is the gradient of the tangent drawn at $x = a$.

2. Calculating the equation of a tangent

Example

Suppose we wish to find the equation of the tangent to

$$f(x) = x^3 - 3x^2 + x - 1$$

at the point where $x = 3$.

When $x = 3$ we note that

$$f(3) = 3^3 - 3.3^2 + 3 - 1 = 27 - 27 + 3 - 1 = 2$$

So the point of interest has coordinates $(3, 2)$.

The next thing that we need is the gradient of the curve at this point. To find this, we need to differentiate $f(x)$:

$$f'(x) = 3x^2 - 6x + 1$$

We can now calculate the gradient of the curve at the point where $x = 3$.

$$f'(3) = 3.3^2 - 6.3 + 1 = 27 - 18 + 1 = 10$$

So we have the coordinates of the required point, $(3, 2)$, and the gradient of the tangent at that point, 10.

What we want to calculate is the equation of the tangent at this point on the curve. The tangent must pass through the point and have gradient 10. The tangent is a straight line and so we use the fact that the equation of a straight line that passes through a point (x_1, y_1) and has gradient m is given by the formula

$$\frac{y - y_1}{x - x_1} = m$$

Substituting the given values

$$\frac{y - 2}{x - 3} = 10$$

and rearranging

$$\begin{aligned} y - 2 &= 10(x - 3) \\ y - 2 &= 10x - 30 \\ y &= 10x - 28 \end{aligned}$$

This is the equation of the tangent to the curve at the point $(3, 2)$.

Key Point

The equation of a straight line that passes through a point (x_1, y_1) and has gradient m is given by

$$\frac{y - y_1}{x - x_1} = m$$

Example

Suppose we wish to find points on the curve $y(x)$ given by

$$y = x^3 - 6x^2 + x + 3$$

where the tangents are parallel to the line $y = x + 5$.

If the tangents have to be parallel to the line then they must have the same gradient. The standard equation for a straight line is $y = mx + c$, where m is the gradient. So what we gain from looking at this standard equation and comparing it with the straight line $y = x + 5$ is that the gradient, m, is equal to 1. Thus the gradients of the tangents we are trying to find must also have gradient 1.

We know that if we differentiate $y(x)$ we will obtain an expression for the gradients of the tangents to $y(x)$ and we can set this equal to 1. Differentiating, and setting this equal to 1 we find

$$\frac{dy}{dx} = 3x^2 - 12x + 1 = 1$$

from which

$$3x^2 - 12x = 0$$

This is a quadratic equation which we can solve by factorisation.

$$3x^2 - 12x = 0$$
$$3x(x - 4) = 0$$
$$3x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

Now having found these two values of x we can calculate the corresponding y coordinates. We do this from the equation of the curve: $y = x^3 - 6x^2 + x + 3$.

when $x = 0$: $y = 0^3 - 6.0^2 + 0 + 3 = 3$.

when $x = 4$: $y = 4^3 - 6.4^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$.

So the two points are $(0, 3)$ and $(4, -25)$

These are the two points where the gradients of the tangent are equal to 1, and so where the tangents are parallel to the line that we started out with, i.e. $y = x + 5$.

Exercise 1

1. For each of the functions given below determine the equation of the tangent at the points indicated.

a) $f(x) = 3x^2 - 2x + 4$ at $x = 0$ and 3.

b) $f(x) = 5x^3 + 12x^2 - 7x$ at $x = -1$ and 1.

c) $f(x) = xe^x$ at $x = 0$.

d) $f(x) = (x^2 + 1)^3$ at $x = -2$ and 1.

e) $f(x) = \sin 2x$ at $x = 0$ and $\frac{\pi}{6}$.

f) $f(x) = 1 - 2x$ at $x = -3$, 0 and 2.

2. Find the equation of each tangent of the function $f(x) = x^3 - 5x^3 + 5x - 4$ which is parallel to the line $y = 2x + 1$.

3. Find the equation of each tangent of the function $f(x) = x^3 + x^2 + x + 1$ which is perpendicular to the line $2y + x + 5 = 0$.

3. The equation of a normal to a curve

In mathematics the word 'normal' has a very specific meaning. It means 'perpendicular' or 'at right angles'.

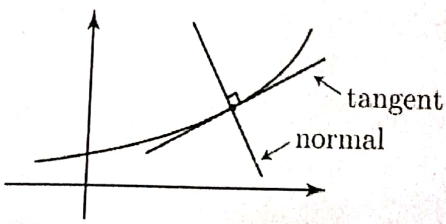


Figure 2. The normal is a line at right angles to the tangent.

If we have a curve such as that shown in Figure 2, we can choose a point and draw in the tangent to the curve at that point. The normal is then at right angles to the curve so it is also at right angles (perpendicular) to the tangent.

We now find the equation of the normal to a curve. There is one further piece of information that we need in order to do this. If two lines, having gradients m_1 and m_2 respectively, are at right angles to each other then the product of their gradients, $m_1 m_2$, must equal -1.

Key Point

If two lines, with gradients m_1 and m_2 are at right angles then $m_1 m_2 = -1$

Example

Suppose we wish to find the equation of the tangent and the equation of the normal to the curve

$$y = x + \frac{1}{x}$$

at the point where $x = 2$.

First of all we shall calculate the y coordinate at the point on the curve where $x = 2$:

$$y = 2 + \frac{1}{2} = \frac{5}{2}$$

Next we want the gradient of the curve at the point $x = 2$. We need to find $\frac{dy}{dx}$.

Noting that we can write y as $y = x + x^{-1}$ then

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Furthermore, when $x = 2$

$$\frac{dy}{dx} = 1 - \frac{1}{4} = \frac{3}{4}$$

This is the gradient of the tangent to the curve at the point $(2, \frac{5}{2})$. We know that the standard equation for a straight line is

$$\frac{y - y_1}{x - x_1} = m$$

With the given values we have

$$\frac{y - \frac{5}{2}}{x - 2} = \frac{3}{4}$$

Rearranging

$$\begin{aligned} y - \frac{5}{2} &= \frac{3}{4}(x-2) \\ 4\left(y - \frac{5}{2}\right) &= 3(x-2) \\ 4y - 10 &= 3x - 6 \\ 4y &= 3x + 4 \end{aligned}$$

So the equation of the tangent to the curve at the point where $x = 2$ is $4y = 3x + 4$.

Now we need to find the equation of the normal to the curve.

Let the gradient of the normal be m_2. Suppose the gradient of the tangent is m_1. Recall that the normal and the tangent are perpendicular and hence $m_1 m_2 = -1$. We know $m_1 = \frac{3}{4}$. So

$$\frac{3}{4} \times m_2 = -1$$

and so

$$m_2 = -\frac{4}{3}$$

So we know the gradient of the normal and we also know the point on the curve through which it passes, $\left(2, \frac{5}{2}\right)$.

As before,

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= m \\ \frac{y - \frac{5}{2}}{x - 2} &= -\frac{4}{3} \end{aligned}$$

Rearranging

$$\begin{aligned} 3\left(y - \frac{5}{2}\right) &= -4(x-2) \\ 3y - \frac{15}{2} &= -4x + 8 \\ 3y + 4x &= 8 + \frac{15}{2} \\ 3y + 4x &= \frac{31}{2} \\ 6y + 8x &= 31 \end{aligned}$$

This is the equation of the normal to the curve at the given point.

Example

Consider the curve $xy = 4$. Suppose we wish to find the equation of the normal at the point $x = 2$. Further, suppose we wish to know where the normal meet the curve again, if it does.

Notice that the equation of the given curve can be written in the alternative form $y = \frac{4}{x}$. A graph of the function $y = \frac{4}{x}$ is shown in Figure 3.

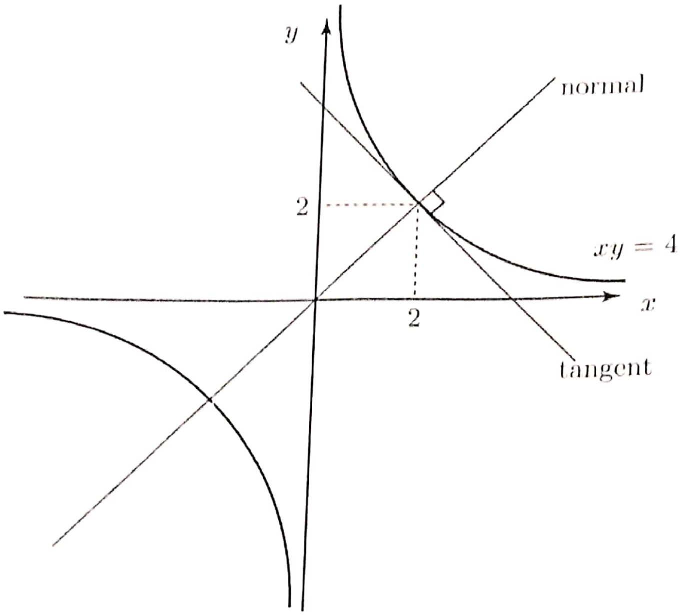


Figure 3. A graph of the curve $xy = 4$ showing the tangent and normal at $x = 2$.

From the graph we can see that the normal to the curve when $x = 2$ does indeed meet the curve again (in the third quadrant). We shall determine the point of intersection. Note that when $x = 2$, $y = \frac{4}{2} = 2$.

We first determine the gradient of the tangent at the point $x = 2$. Writing

$$\begin{aligned} y &= \frac{4}{x} \\ &= 4x^{-1} \end{aligned}$$

and differentiating, we find

$$\begin{aligned} \frac{dy}{dx} &= -4x^{-2} \\ &= -\frac{4}{x^2} \end{aligned}$$

Now, when $x = 2$ $\frac{dy}{dx} = -\frac{4}{4} = -1$.

So, we have the point $(2, 2)$ and we know the gradient of the tangent there is -1. Remember that the tangent and normal are at right angles and for two lines at right angles the product of their gradients is -1. Therefore we can deduce that the gradient of the normal must be $+1$. So, the normal passes through the point $(2, 2)$ and its gradient is 1.

As before, we use the equation of a straight line in the form:

$$\begin{aligned}\frac{y-y_1}{x-x_1} &= m\\ \frac{y-2}{x-2} &= 1\\ y-2 &= x-2\\ y &= x\end{aligned}$$

So the equation of the normal is $y = x$.

We can now find where the normal intersects the curve $xy = 4$. At any points of intersection both of the equations

$$xy = 4 \qquad \text{and} \qquad y = x$$

are true at the same time, so we solve these equations simultaneously. We can substitute $y = x$ from the equation of the normal into the equation of the curve:

$$\begin{aligned}xy &= 4\\ x\cdot x &= 4\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

So we have two values of x where the normal intersects the curve. Since $y = x$ the corresponding y values are also 2 and -2. So our two points are $(2,2)$, $(-2,-2)$. These are the two points where the normal meets the curve. Notice that the first of these is the point we started off with.

Exercise 2

1. For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

a) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

b) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

c) $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$.

d) $f(x) = 3 - x$ at $x = -2$, 0 and 1.

2. Find the equation of each normal of the function $f(x) = \frac{1}{3}x^3 + x^2 + x - \frac{1}{3}$ which is parallel to the line $y = -\frac{1}{4}x + \frac{1}{3}$.

3. Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again.

Answers

Exercise 1

1. a) $y = -2x + 4$, $y = 16x - 23$ b) $y = -16x - 2$, $y = 32x - 22$ c) $y = x$,
d) $y = -300x - 0475$, $y = 24x - 16$, e) $y = 2x$, $y = x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$,
f) $y = 1 - 2x$, $y = 1 - 2x$, $y = 1 - 2x$

2. $y = 2x - \frac{95}{27}$, $y = 2x - 13$

3. $y = 2x + 2$, $y = 2x + \frac{22}{27}$

Exercise 2

1. a) At $x = 0$: $y = 3x + 1$, $y = -\frac{1}{3}x + 1$, At $x = 4$: $y = 11x - 15$, $y = -\frac{1}{11}x + \frac{323}{11}$
b) At $x = -1$: $y = x + 8$, $y = -x + 6$, At $x = 1$: $y = x$, $y = -x + 2$
c) At $x = \frac{\pi}{4}$: $y = 2x + 1 - \frac{\pi}{2}$, $y = -\frac{1}{2}x + 1 + \frac{\pi}{8}$
d) At $x = -2$: $y = 3 - x$, $y = x + 7$, At $x = 0$: $y = 3 - x$, $y = x + 3$,
At $x = 1$: $y = 3 - x$, $y = x + 1$

2. $y = -\frac{1}{4}x + \frac{9}{4}$, $y = -\frac{1}{4}x - \frac{49}{12}$

3. $\frac{21}{5}$

CURVATURE AND RADIUS OF CURVATURE

5.1 Introduction:

Curvature is a numerical measure of bending of the curve. At a particular point on the curve , a tangent can be drawn. Let this line makes an angle Ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length s.

∴ Curvature at P = $\left|\frac{d\Psi}{ds}\right|$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

5.2

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+y_1^2\right)^{3/2}}{|y_2|} \text{ (When tangent is parallel to x – axis)}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \text{ (When tangent is parallel to y – axis)}$$

- **Radius of curvature of parametric curve:**

$x = f(t), y = g(t)$

$$\rho = \frac{\left(x'^2+y'^2\right)^{3/2}}{|x'y''-y'x''|}, \text{ where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 1 Find the radius of curvature at any pt of the cycloid

$x = a\,(\theta + \text{Sin}\,\theta),\ \ y = a\,(1- \cos\theta)$

Solution: $x' = \frac{dx}{d\theta} = a\,(1+\cos\theta)$ and $y' = \frac{dy}{d\theta} = a\sin\theta$

$x'' = \frac{d^2x}{d\theta^2} = -a\sin\theta$ and $y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$

Now $\rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|} = \frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)\cos\theta + a^2\sin^2\theta}$

$$= \frac{a(1+\cos^2\theta+2\cos\theta+\sin^2\theta)^{3/2}}{\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{a(2+2\cos\theta)^{3/2}}{1+\cos\theta}$$

$$= 2\sqrt{2}\,a\sqrt{1+\cos\theta}$$

$$= 2\sqrt{2}\,a\sqrt{2\frac{\cos^2\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Example 2 Show that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ($x = a\cos^3\theta$, $y = a\sin^3\theta$) is equal to three times the lenth of the perpendicular from the origin to the tangent.

Solution : $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta = x'$

$$\frac{dy}{d\theta} = -3a\sin^2\theta\cos\theta = y'$$

$$x'' = \frac{d^2x}{d\theta^2} = \frac{d}{d\theta}(-3a\cos^2\theta\sin\theta)$$

$$= -3a[-2\cos\theta\sin^2\theta+\cos^3\theta]$$

$$= 6a\cos\theta\sin^2\theta - 3a\cos^3\theta$$

$$y'' = \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(3a\sin^2\theta\cos\theta)$$

$$= 3a(2\sin\theta\cos^2\theta - \sin^3\theta)$$
$$= 6a\sin\theta\cos^2\theta - 3a\sin^3\theta$$

Now $\rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}$

$$= \frac{(9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta)^{3/2}}{|(-3a\cos^2\theta\sin\theta)(6a\sin\theta\cos^2\theta-3a\sin^3\theta)-3a\sin^2\theta\cos\theta(6a\cos\theta\sin^2\theta-3a\cos^3\theta)|}$$

$$= \frac{\left[9a^2\cos^2\sin^2\theta\,(\cos^2\theta+\sin^2\theta)\right]^{3/2}}{\left|-18a^2\sin^2\theta\cos^4\theta+9a^2\cos^2\theta\sin^4\theta-18a^2\sin^4\theta\cos^2\theta+9a^2\sin^2\theta\cos^4\theta\right|}$$

$$= \frac{9^{3/2}(a\cos\theta\sin\theta)^3}{\left|-9a^2\sin^2\theta\cos^4\theta-9a^2\cos^2\theta\sin^4\theta\right|}$$

$$= \frac{(9)^{3/2}(a\cos\theta\sin\theta)^3}{9a^2\cos^2\theta\sin^2\theta\,(\cos^2\theta+\sin^2\theta)}$$

$\Rightarrow \rho = 3a\sin\theta\cos\theta$ ....(1)

The equation of the tangent at any point on the curve is

$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$

$\Rightarrow x\sin\theta + y\cos\theta - a\sin\theta\cos\theta = 0$ ....(2)

∴ The length of the perpendicular from the origin to the tangent (2) is

$$p = \frac{|0.\sin\theta+0.\cos\theta-a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}}$$

$= a\sin\theta\cos\theta$ ....(3)

Hence from (1) & (3), $\rho = 3p$

Example 3 If ρ & ρ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ prove that

$$\left(\rho^{2/3}+\rho'^{2/3}\right)(ab)^{2/3} = a^2+b^2$$

Solution: Parametric equation of the ellipse is

$x = a\cos\theta,\quad y = b\sin\theta$

$x' = -a\sin\theta,\quad y' = b\cos\theta$

$x'' = -a\cos\theta,\quad y'' = -b\sin\theta$

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|} = \frac{(a^2\sin^2\theta+b^2\cos^2 b)^{3/2}}{|(-a\sin\theta)(-b\sin\theta)-(b\cos\theta)(-a\cos\theta)|}$$

$$= \frac{(a^2 sin^2\theta + b^2 cos^2\theta)^{3/2}}{ab} \qquad ......(1)$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by replacing θ by $\theta + \frac{\pi}{2}$ in (1).
Let it be denoted by ρ'. Then

$$\therefore \rho' = \frac{(a^2 sin^2\theta + b^2 sin^2\theta)^{3/2}}{ab}$$

$$\therefore \rho^{2/3} + \rho'^{2/3} = \frac{a^2 sin^2\theta + b^2 cos^2\theta}{(ab)^{2/3}} + \frac{a^2 cos^2\theta + b^2 cos^2\theta}{(ab)^{2/3}}$$

$$= \frac{a^2 + b^2}{(ab)^{2/3}}$$

$$\Rightarrow (ab)^{2/3}\left(\rho^{2/3} + \rho'^{2/3}\right) = a^2 + b^2$$

Example 4 Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $\frac{125}{16}$.

Solution: $y = 2\sqrt{2}\sqrt{x}$

$$y_1 = \frac{\sqrt{2}}{\sqrt{x}} \quad , \quad y_2 = \frac{-1}{\sqrt{2}x^{3/2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} = (1 + \frac{2}{x})^{3/2}.\sqrt{2}\, x^{3/2} = \sqrt{2}\,(x+2)^{3/2}$$

Given $\rho = \frac{12.5}{16}$ $\therefore (x + 2)^{3/2} = \frac{125}{16\sqrt{2}} = \left(\frac{5}{2\sqrt{2}}\right)^3$.

$$\therefore (x + 2)^{3/2} = \frac{5}{2\sqrt{2}}$$

$$\Rightarrow x + 2 = \frac{25}{8} \quad \Rightarrow x = \frac{9}{8}$$

$$\Rightarrow y^2 = 8\left(\frac{9}{8}\right) \text{ i.e. } y = 3,-3$$

Hence the points at which the radius of curvature is $\frac{125}{16}$ are $(9, \pm 3)$.

Example 5 Find the radius of curvature at any point of the curve y = C cos h (x/c)

Solution: $y_1 = \frac{c}{c}$ Sin h $\frac{x}{c}$ = Sin h $\left(\frac{x}{c}\right)$

$y_2 = \frac{1}{c}$ cos h $\frac{x}{c}$

Now, $\rho = \frac{(1+y_1{}^2)^{3/2}}{y_2}$

$= \frac{\left(1+Sin\ h^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cos h\frac{x}{c}}$

$=$ C cos $h^2\left(\frac{x}{c}\right)$

$\Rightarrow \rho = \frac{1}{c}y^2$

Example 6 For the curve $y = \frac{ax}{a+x}$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

where ρ is the radius of curvature of the curve at its point (x, y)

Solution: Here $y = \frac{ax}{a+x}$

$\Rightarrow y_1 = \frac{(a+x)a - ax\,(1)}{(a+x)^2}$

$= \frac{a^2}{(a+x)^2}$

$\therefore y_2 = \frac{-2a^2}{(a+x)^3}$

Now, $\rho = \frac{\left(1+y^{1^2}\right)^{3/2}}{y_2}$

$= \left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{(-2a^2)}$

$\therefore \rho^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{(-2)^{2/3}\, a^{4/3}}$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[1+\frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{2^{2/3}\,a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}}$$

$$= \frac{1}{a^2}\left[1+\frac{a^4}{(a+1)^4}\right](a+x)^2$$

$$= \frac{1}{a^2}\left[(a+x)^2+\frac{a^4}{(a+x)^2}\right]$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

Example 7 Find the curvature of $x = 4\cos t$, $y = 3\sin t$. At what point on this ellipse does the curvature have the greatest & the least values? What are the magnitudes?

Solution: $\rho = \dfrac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}$

Now, $x' = -4\sin t \Rightarrow x'' = -4\cos t$

$y' = -3\cos t \Rightarrow x'' = -3\sin t$

$$\therefore \rho = \frac{(16\sin^2 t+9\cos t^2 t)^{3/2}}{-4\sin t\,(-3\sin t)-3\cos t\,(-4\cos t)}$$

$$= \frac{1}{12}(9\cos t^2 t + 16\sin^2 t)^{3/2}$$

$$\Rightarrow (\rho.12)^{2/3} = 9\cos t^2 t + 16\sin^2 t$$

Now, curvature is the reciprocal of radius of curvature. Curvature is maximum & minimum when ρ is minimum and maximum respectively . For maximum and minimum values;

$$\frac{d}{dt}(16\sin^2 t + 9\cos^2 t) = 0$$

$$\Rightarrow 32\sin t\cos t + 18\cos t(-\sin t) = 0$$

$\Rightarrow$ $4 \sin t \cos t = 0$

$\Rightarrow t = 0 \ \& \ \frac{\pi}{2}$

At $t = 0$ ie at (4,0)

$(12\rho)^{2/3} = 9$

$\Rightarrow 12\rho = 9^{3/2}$

$\Rightarrow \rho = \frac{9}{4} \quad \therefore \frac{1}{\rho} = \frac{4}{9}$

Similarly, at $t = \frac{\pi}{2}$ ie at (0,3)

$(12\rho)^{2/3} = 16$

$12\rho = 4^3$

$\rho = 16/3 \quad \therefore \frac{1}{\rho} = \frac{3}{16}$

Hence, the least value is $\frac{3}{16}$ and the greatest value is $\frac{4}{9}$

Example 8 Find the radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the coordinate axes.

Solution: On differentiating the given , we get

$$\frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{by}{ax}} \qquad ....(1)$$

The curve touches the x-axis if $\frac{dy}{dx} = 0$ or $y = 0$

When $y = 0$, we have $x = a$ (from the given eq^n)

$\Rightarrow$ given curve touches x – axis at (a,0)

The curve touches y – axis if $\frac{dx}{dy} = 0$ or $x = 0$

When $x = 0$, we have $y = b$

$\Rightarrow$ Given curve touches y-axis at (o, b)

$$\frac{d^2y}{dx} = \sqrt{\frac{b}{a}}\left\{\sqrt{\frac{b}{a}} \cdot \frac{1}{2x} - \frac{1}{2}\sqrt{\frac{y}{x}}\right\} \qquad \{\text{from (1)}\}$$

At (a,0), $\frac{d^2y}{dx^2} = \frac{1}{2a}\frac{b}{a} = \frac{b}{2a^2}$

$\therefore$ At (a,o), $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = (1+0)^{3/2}\,\frac{2a^2}{b} = \frac{2a^2}{b}$

At (o,b), $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{2b^2}{a}$

5.3 Radius of curvature of Polar curves r = f (θ):

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{2r_1^2+r^2-rr_2} \qquad \left(where\ r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}\right)$$

Example 9 Prove that for the cardioide r = a (1 + cos θ),

$\frac{\rho^2}{r}$ is const.

Solution: Here r = a (1+ cos θ)

$$\Rightarrow r_1 = -\text{a Sin }\theta \text{ and } r_2 = -\text{a cos }\theta$$

$$\therefore r^2 + r_1^2 = a^2\left[(1+\cos\theta)^2 + \sin^2\theta\right] = 2a^2\,(1+\cos\theta)$$

$$r^2 + 2r_1^2 - rr^2 = a^2[(1+\cos\theta)^2 + 2\sin^2\theta + \cos\theta(1+\cos\theta)]$$

$$= 3a^2\,(1+\cos\theta)$$

$$\therefore \rho^2 = \frac{(r^2+r_1^2)^3}{(r^2+2r_1^2-rr_2)^2} = \frac{8a^6(1+\cos\theta)^3}{9a^4\,(1+\cos\theta)^2} = \frac{8}{9}a^2\,(1+\cos\theta)$$

$$\Rightarrow \rho^2 = \frac{8a}{9}\,r$$

$\therefore \frac{\rho^2}{r} = \frac{8a}{9}$ which is a constant.

Example 10 Show that at the point of intersection of the curves r = a θ and r θ = a, the curvatures are in the ratio 3:1 ($0 < \theta < 2\pi$)

Solution: The points of intersection of curves $r = a\,\theta$ & $r\,\theta = a$ are given by $a\,\theta^2 = a$ or $\theta = \pm 1$

Now for the curve $r = a\,\theta$ we have $r_1 = a$ and $r_2 = 0$

$$\therefore \text{At } \theta = \pm 1,\ \rho = \left[\frac{(r^2+r_1^2)^{3/2}}{2a^2+a^2\theta^2-0}\right]_{\theta=\pm 1} = \frac{a\,(2\sqrt{2})}{3} = \rho_1$$

For the curve $r\,\theta = a$,

$$r_1 = \frac{-a}{\theta^2} \quad \text{and} \quad r_2 = \frac{2a}{\theta^3}$$

$$\text{At } \theta = \pm 1,\ \rho = \left[\frac{\left(\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right)^{3/2}}{\frac{2a^2}{\theta^4}+\frac{a^2}{\theta^2}-\frac{2a^2}{\theta^4}}\right]_{\theta=\pm 1,} = \left[a\,\frac{(1+\theta^2)^{3/2}}{\theta^4}\right]_{\theta=\pm 1}$$

$$= 2a\sqrt{2} = \rho_2$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{2a\,\sqrt{2}}{2a\,\sqrt{2/3}} = \frac{3}{1}$$

$$\therefore \rho_2 : \rho_1 = 3 : 1$$

Example 11 Find the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\,\theta$

Solution: $r^m = a^m \cos m\,\theta$

$$\Rightarrow m\log r = m\log a + \log\cos m\,\theta$$

$$\Rightarrow \frac{m}{r}\,r_1 = -\,m\,\frac{\sin m\theta}{\cos m\theta} \quad \text{(on differentiating w.r.t. } \theta)$$

$$\Rightarrow r_1 = -\,r\tan m\,\theta \qquad \ldots\ldots.(1)$$

$$\text{Now } r_2 = -\,(r_1 \tan m\,\theta + rm\sec^2 m\,\theta)$$

$$= r\tan^2 m\,\theta - rm\sec^2 m\,\theta \quad \text{(from (1))}$$

$$\therefore \rho = \frac{(r^2+r^2\tan^2 m\theta)^{3/2}}{r^2+2r^2\tan^2 m\theta-r^2\tan^2 m\theta+r^2 m\sec^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta+r^2 m\sec^2 m\theta} \qquad = \frac{r}{m+1}\sec m\theta$$

Example 12 Show that the radius of curvature at the point (r, θ) of the curve $r^2\cos 2\theta = a^2$ is $\frac{r^3}{a^2}$

Solution: $r^2 = a^2\sec 2\theta$

$$\Rightarrow 2rr_1 = 2a^2\sec 2\theta\tan 2\theta$$
$$\Rightarrow r_1 = r\tan 2\theta$$

and $r_2 = 2r\sec^2\theta + r_1\tan 2\theta$

$$= 2r\sec^2 2\theta + r\tan^2 2\theta \quad (\because r = r\tan 2\theta)$$

Now $\rho = \frac{(r^1+r_1^2)^{3/2}}{2r_1^2+r^2-rr_2} \Rightarrow \rho = \frac{((r^2+r^2\tan^2 2\theta))^{3/2}}{2r^2\tan^2 2\theta+r^2-r^2(2\sec^2 2\theta+\tan^2 2\theta)}$

$$= \frac{(r^2\sec^2 2\theta)^{3/2}}{r^2(2\tan^2 2\theta+1-2\sec^2 2\theta-\tan^2 2\theta)}$$

$$= \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta}$$

$$= r\sec 2\theta$$

$$= r\,.\frac{r^2}{a^2} = \frac{r^3}{a^2}$$

5.4 Radius of curvature at the origin by Newton's method

It is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent there.

When x-axis is the tangent, then

$$\rho = \lim_{x\to 0}\frac{x^2}{2y}$$

When y- axis is the tangent, then

$$\rho = \lim_{x\to 0} \frac{y^2}{2x}$$

Example13 Find the radius of curvature at the origin of the curve $x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$

Solution: Tangent is x = 0 ie y–axis,

$$\rho = \lim_{y\to 0} \frac{y^2}{2x}$$

Dividing the given equation by 2x, we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{+xy}{2x} \frac{-y^2}{2x} + \frac{2x}{2x} = 0$$

$$x^3\left(\frac{y}{2x}\right) - xy\left(\frac{y^2}{2x}\right) + xy + x\left(\frac{y}{2x}\right) - \left(\frac{y^2}{2x}\right) + 1 = 0$$

Taking limit $y \to 0$ on both the sides , we get $\rho = 1$

Exercise 5A

1. Find the radius of curvatures at any point the curve
   $y = 4\sin x - \sin 2x$ at $x = \frac{\pi}{2}$

   Ans $\rho = \frac{1}{4}(5)^{3/2}$

2. If ρ_1, ρ_2 are the radii of curvature at the extremes of any chord of the cardioide $r = a(1 + \cos\theta)$ which passes through the pole, then

   $\rho_1^1 + \rho_2^2 = \frac{16a^2}{9}$

3 Find the radius of curvature of $y^2 = x^2(a+x)(a-x)$ at the origin

Ans. $a\sqrt{2}$

4. Find the radius of curvature at any point 't' of the curve
   $x = a(\cos t + \log\tan t/2)$, $y = a\sin t$

   Ans. $a\cos t$

5. Find the radius of curvature at the origin, for the curve
$2x^3 - 3x^2y + 4y^3 + y^2 - 3x = 0$

Ans. $\rho = 3/2$

6. Find the radius of curvature of $y^2 = \frac{4a^2(2a-x)}{x}$ at a point where the curve meets x – axis

Ans. $\rho = a$

7. Prove the if ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus rectum is l then
$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}$

8. Find the radius of curvature to the curve $r = a\,(1 + \cos\theta)$ at the point where the tangent is parallel to the initial line.

Ans. $\rho = \frac{2}{\sqrt{3}} \cdot a$

9. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho = \frac{a^2b^2}{p^3}$ where p is the perpendicular distance from the centre on the tangent at (x,y).

UNIT – II

GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS

Curvature:

At each point on a curve, with equation y=f(x), the tangent line turns at a certain rate. A measure of this rate of turning is the curvature

$$K = \frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}$$

Radius of curvature in Cartesian form:

If the curve is given in Cartesian coordinates as *y(x)*, then the radius of curvature is

$\rho = (1 + [y']^2)^{(3/2)}/y''$ where $y' = \frac{dy}{dx}$, $y'' = (d^2 y)/(dx^2)$.

Radius of curvature in Parametric form:

If the curve is given parametrically by functions *x(t)* and *y(t)*, then the radius of curvature is

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - x''y'}, \quad x' = \frac{dx}{dt}, x'' = \frac{d^2x}{dt^2}, y' = \frac{dy}{dt}, y'' = \frac{d^2y}{dt^2}$$

Examples:

1. Find the radius of the curvature at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = 1$.

Solution: $\sqrt{x} + \sqrt{y} = 1$

Differentiating w. r. t x ,we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0 \quad . \quad y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

At $\left(\frac{1}{4}, \frac{1}{4}\right)$, $y' = -1.$

$$y'' = -[(\sqrt{x}\ 1/(2\sqrt{y})\ y' - \sqrt{y}\ 1/(2\sqrt{x}))/x]$$

At $\left(\frac{1}{4}, \frac{1}{4}\right)$, $y'' = -[(1/2\ \ 1/(2\ 1/2)\ (-1) - 1/2\ \ 1/(2\ 1/2))/(1/4)] = 4$.

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{1}{\sqrt{2}}$$

2. Show that the radius of the curvature at any point of the curve $y = c\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$.

Solution: $y = c\cosh\left(\frac{x}{c}\right)$

Differentiating y w. r. t x we get

$y' = \sinh\left(\frac{x}{c}\right)$

$y'' = 1/c\ \ \cosh(x/c)$

$$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = c\cosh^2\left(\frac{x}{c}\right) = \frac{y^2}{c}.$$

3. Find the radius of the curvature of the curve y = x^2(x-3) at the points where the tangent is parallel to the x – axis.

Solution: y = x^2(x-3)

Differentiating y w. r. t x we get

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

The points at which the tangent parallel to the x – axis can be found by equating y' to zero.

i.e., $3x^2 - 6x = 0 \Rightarrow x = 0, x = 2.$

At x = 0,y" = -6. At x = 2, y" = 6.

Therefore at x = 0 and x = 2, $\rho = \frac{1}{6}$.

4. Prove that the radius of the curvature of the curve at any point of the cycloid $x = a(t + \sin t), y = a(1 + \cos t)$ is $\frac{4a\cos t}{2}$.

Solution: We have $x = a(t + \sin t), y = a(1 + \cos t)$.

Therefore $\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a\sin t$.

Now $\frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2}}{2\cos^2\frac{t}{2}} = \frac{\tan t}{2}$.

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\tan t}{2}\right) = \left\{\frac{d}{dt}\left(\frac{\tan t}{2}\right)\right\}\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{t}{2}\frac{1}{2a(1+\cos t)} = \frac{1}{4a}\sec^4\frac{t}{2}$.

Hence $\rho = \frac{\left(1+\tan^2\frac{t}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}\sec^4\frac{t}{2}} = \frac{4a\cos t}{2}$.

Centre and Circle of curvature:

Let the equation of the curve be y = f(x). let P be the given point (x,y) on this curve and Q the point (x+Δx,y+Δy) in the neighborhood of P. let N be the point of intersection of the normals at P and Q. As Q→P, suppose N→C. Then C is the centre of curvature of P. The circle whose centre C and radius ρ is called the circle of curvature. The co-ordinates of the centre of curvature is denoted as $(\bar{x}, \bar{y})$.

where $\bar{x} = x - (y'(1+(y')^2))/y''$, $\bar{y} = y + ((1+(y')^2))/y''$.

Equation of the circle of curvature:

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature and ρ be the radius of curvature at any point (x,y) on a curve, then the equation of the circle of curvature at that point is

$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

Examples:

1. Find the centre of curvature of the curve $a^2y = x^3$.

   Solution: $a^2y = x^3$

   $\frac{dy}{dx} = \frac{3x^2}{a^2}$ and $\frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$$\overline{x} = x - \frac{x}{2}\left(1 + \frac{9x^4}{a^4}\right) = \frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right]$$

$$\overline{y} = \frac{x^3}{a^2} + \frac{\left[1 + \frac{9x^4}{a^4}\right]}{\frac{6x}{a^2}} = \frac{5x^3}{2a^2} + \frac{a^2}{6x}$$

Therefore the required centre of curvature is $\left(\frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right], \frac{5x^3}{2a^2} + \frac{a^2}{6x}\right)$.

2. Find the centre of curvature of $y = x^2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

Solution: $y' = 2x$, $y'' = 2$.

At $\left(\frac{1}{2}, \frac{1}{4}\right)$, $y' = 1$, $y'' = 2$.

Therefore $\overline{x} = \frac{1}{2} - \frac{(1+1)}{2} = -\frac{1}{2}, \overline{y} = \frac{1}{4} + 1 = \frac{5}{4}$.

Therefore the required centre of curvature is $\left(-\frac{1}{2}, \frac{5}{4}\right)$.

3. Find the centre of curvature of the curve $xy = a^2$ at (a,a).

Solution: $y' = -a^2/x^2$, $y'' = 2a^2 x^{(-3)}$. At (a,a) $y' = -1$, $y'' = \frac{2}{a}$

Therefore $\overline{x} = a + \frac{2}{2/a} = 2a, \overline{y} = a + \frac{2}{2/a} = 2a.$

The required centre of curvature is $(2a, 2a)$.

4. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Solution: $x^3 + y^3 = 3axy$

$3x^2 + 3y^2 y' = 3a(xy' + y)$

$$y' = \frac{ay - x^2}{y^2 - ax}$$

y' at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is -1.

$y'' = ((y'^2 - ax)(ay'' - 2x) - (ay - x^2)(2yy' - a))/(y^2 - ax)^2$

y'' at $(3a/2, 3a/2) = (-32)/3a$

$$\rho = \frac{2\sqrt{2}(3a)}{32}$$

$$\bar{x} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}$$

$$\bar{y} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}$$

The circle of curvature is $\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \frac{9a^2}{128}$

5. Find the circle of curvature at the point (2,3) on $\frac{x^2}{4} + \frac{y^2}{9} = 2$

Solution: $\frac{2x}{4} + \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{-9x}{4y} \Rightarrow y'(2,3) = \frac{-3}{2}$

$y'' = (-9(y - xy'))/(4y^2)$, y'' at $(2,3) = (-3)/2$

$$\rho = \frac{13^{\frac{3}{2}}}{12}, \quad \bar{x} = 2 - \frac{(-3/_2)(1 + 9/_4)}{\frac{-3}{2}} = \frac{-5}{4},$$

$$\bar{y} = 3 + \frac{(1 + 9/_4)}{\frac{-3}{2}} = \frac{5}{6}$$

The circle of curvature is $\left(x + \frac{5}{4}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{13^3}{12^2}$

Evolute and Involute

Evolute: Evolute of the curve is defined as the locus of the centre of curvature for that curve.

Involute : If C' is the evolute of the curve C then C is called the involute of the curve C'.

Procedure to find the evolute:

Let the given curve be f(x,y,a,b) = 0. (1)

Find y' and y'' at the point P.

Find the centre of curvature $(\bar{x}, \bar{y})$. Using $\bar{x} = x - (y'(1 + y'^2))/y''$,
$\bar{y} = y + ((1 + y'^2))/y''$. (2)

Eliminate x, y from (1), (2) we get $f(\bar{x}, \bar{y}, a, b) = 0$. (3)

Equation (3) is the required evolute.

Examples:

1. Show that the evolute of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ is another cycloid given by $x = a(\theta - \sin\theta), y - 2a = a(1 + \cos\theta)$.

Solution: $\frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 + \cos\theta)} = \frac{\tan\theta}{2}$$

$y'' = d/d\theta\ (\tan\theta/2)\ (d\theta)/dx = (\sec^4 \theta/2)/4a$

$$\bar{x} = a(\theta + \sin\theta) - \frac{\frac{\tan\theta}{2\left(1 + \tan^2\frac{\theta}{2}\right)}}{\sec^4\frac{\theta}{2}/4a} = a(\theta + \sin\theta) - 2a\sin\theta = a(\theta - \sin\theta)$$

$$\bar{y} = a(1 - \cos\theta) + \frac{\left(1 + \tan^2\frac{\theta}{2}\right)}{\sec^4\frac{\theta}{2}/4a} = a(1 - \cos\theta) + 4a\cos^2\frac{\theta}{2} = a(1 + \cos\theta) + 2a.$$

$\bar{x} = a(\theta - \sin\theta), \bar{y} - 2a = a(1 + \cos\theta)$.

The locus of $\bar{x}$ and $\bar{y}$ is $x = a(\theta - \sin\theta), y - 2a = a(1 + \cos\theta)$.

2. Prove that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ is a circle $x^2 + y^2 = a^2$.

Solution: $\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$, $\frac{dy}{d\theta} = a\theta\sin\theta$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\cos\theta}{a\theta\sin\theta} = \tan\theta$$

$y'' = 1/(a\theta \cos^3\theta)$,

$$\overline{x} = a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta(1+\tan^2\theta)}{1/a\theta\cos^3\theta} = a\cos\theta,$$

$$\overline{y} = a(\sin\theta - \theta\cos\theta) + \frac{(1+\tan^2\theta)}{1/a\theta\cos^3\theta} = a\sin\theta.$$

Eliminating , $\overline{x}$ *and* $\overline{y}$ we get $\overline{x}^2 + \overline{y}^2 = a^2$.

The evolute of the given curve is $x^2 + y^2 = a^2$.

4.17 Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x - 1}$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.34(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure 4.34(b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

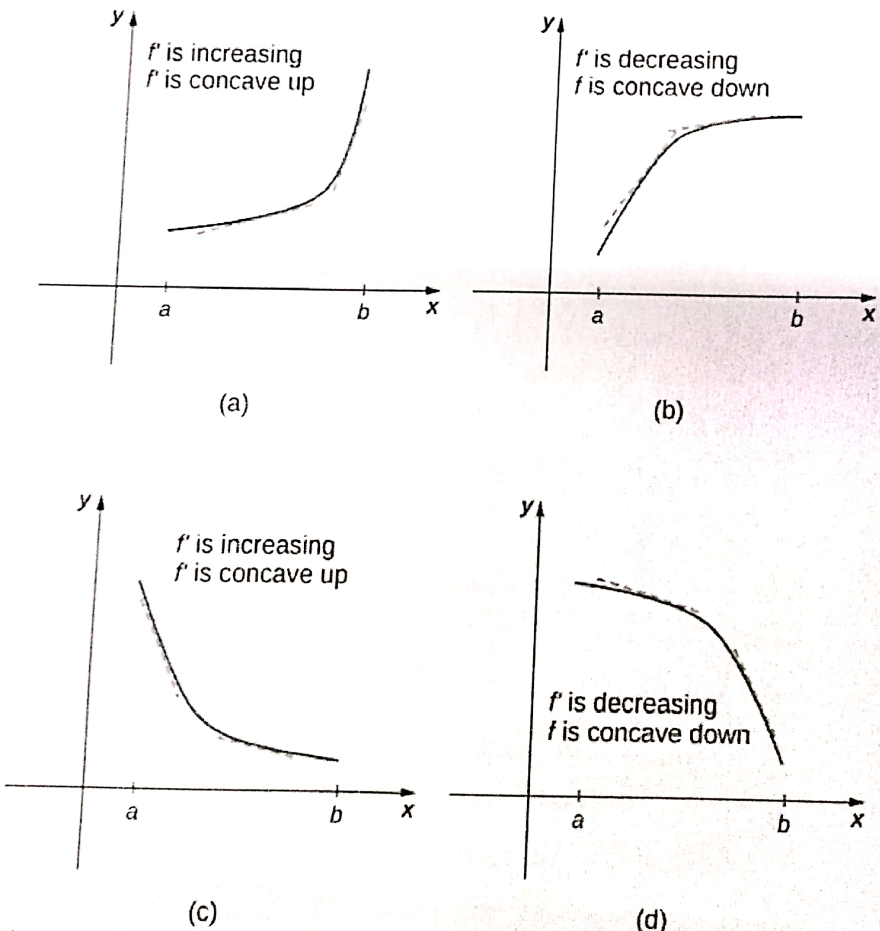


Figure 4.34 (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). (b), (d) Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

Theorem 4.10: Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 4.35). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an inflection point of f.

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f.

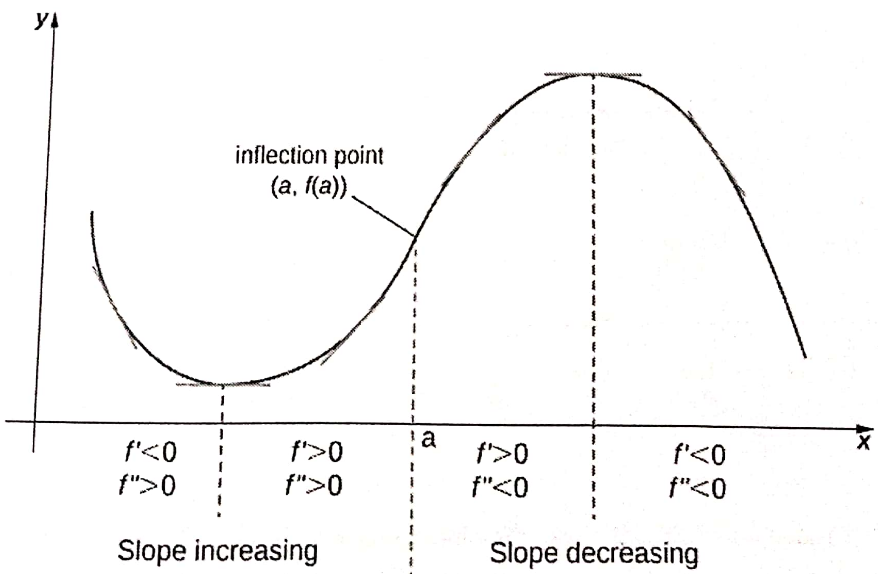


Figure 4.35 Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.19

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	−	f is concave down
$(2, \infty)$	$x = 3$	+	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 4.36 confirms the analytical results.

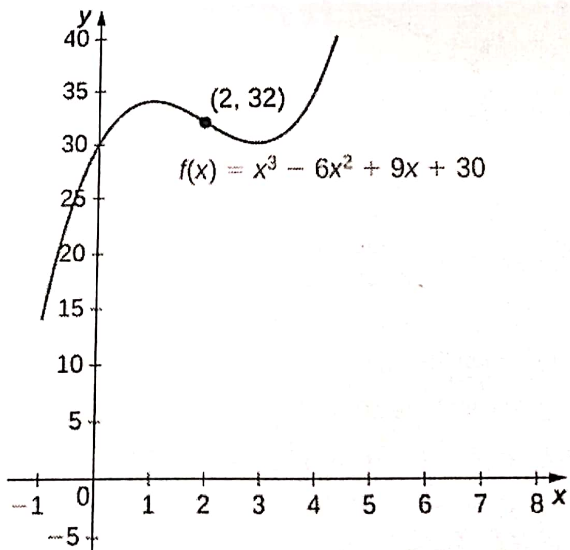


Figure 4.36 The given function has a point of inflection at (2, 32) where the graph changes concavity.

4.18 For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x,$ find all intervals where f is concave up and all intervals where f is concave down.

We now summarize, in Table 4.6, the information that the first and second derivatives of a function f provide about the graph of $f,$ and illustrate this information in Figure 4.37.

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.6 What Derivatives Tell Us about Graphs

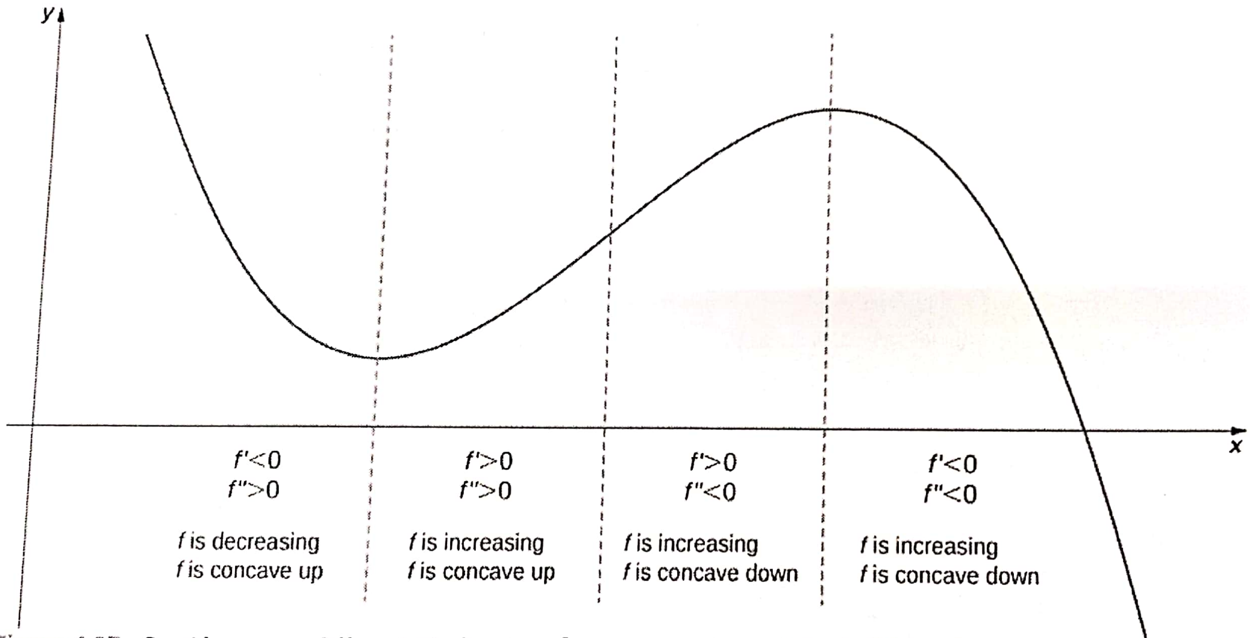


Figure 4.37 Consider a twice-differentiable function f over an open interval $I.$ If $f'(x) > 0$ for all $x \in I,$ the function is increasing over $I.$ If $f'(x) < 0$ for all $x \in I,$ the function is decreasing over $I.$ If $f''(x) > 0$ for all $x \in I,$ the function is concave up. If $f''(x) < 0$ for all $x \in I,$ the function is concave down on $I.$

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing $a.$ Suppose $f''(a) < 0.$ Since f'' is continuous over $I,$ $f''(x) < 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is a decreasing function over $I.$ Since $f'(a) = 0,$ we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

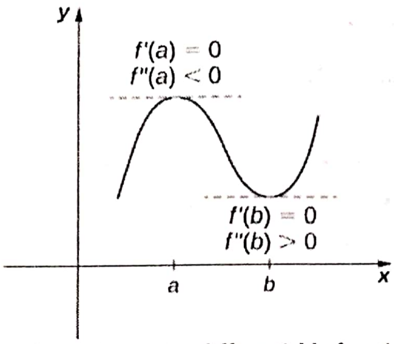


Figure 4.38 Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Theorem 4.11: Second Derivative Test

Suppose $f'(c) = 0$, f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.

iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at x, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.20

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

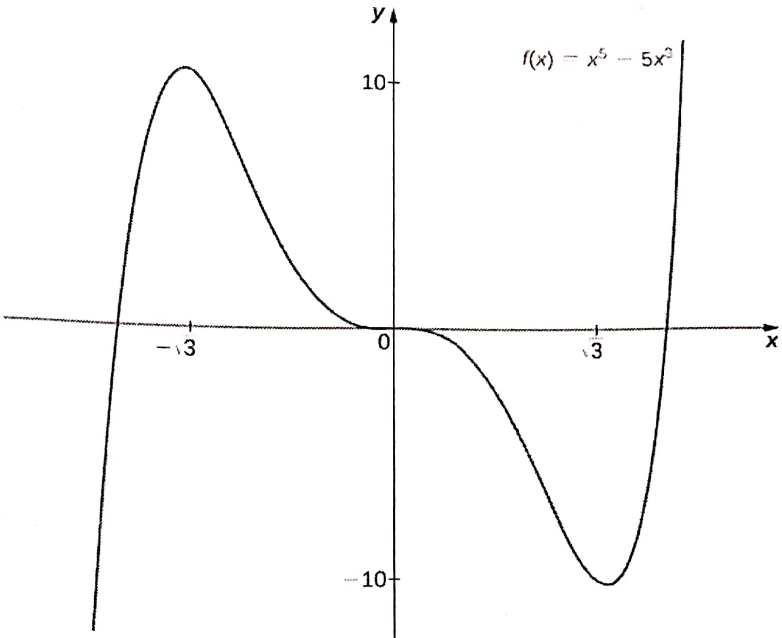


Figure 4.39 The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

4.19 Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Unit-3

Course Name: Calculus, Geometry & Differential Equations
Semester: I
Course Instructor: Dr. Amiya Biswas
Session: 2021-2022
Affiliation: Department of Mathematics, Durgapur Government College, West Bengal, India

Tracing of curves

For the given equation of a curve, though the nature of the curve can be studied by investing properties such as continuity and differentiability at different points of the curve, a diagrammatic representation of the curve often helps the readers to understand these properties in a better way. In this chapter, we first discuss the general rules to be followed to trace the curve of a given equation, given in Cartesian, parametric and polar form.

(a) Tracing of curves in cartesian form

Let $y = f(x)$ or $f(x, y) = 0$ be the equation of a given curve in Cartesian form.

Symmetry of a curve about x-axis: Now, if the curve be symmetric about x-axis, then both the points (x, y) and $(x, -y)$ lie on the curve (shown in Fig. 1), i.e., $f(x, y) = 0$ and $f(x, -y) = 0$, and hence, we may write $f(x, -y) = \pm f(x, y)$. In other words, if by replacing y by $-y$, the curve remains unchanged, then the curve is symmetrical about the x-axis.

Symmetry of a curve about y-axis: If the curve be symmetric about y-axis, then both the points (x, y) and $(-x, y)$ lie on the curve (shown in Fig. 1), i.e., $f(x, y) = 0$ and $f(-x, y) = 0$, and hence, we may write $f(-x, y) = \pm f(x, y)$. In other words, if by replacing x by $-x$, the curve remains unchanged, then the curve is symmetrical about the y-axis.

Explanation: Let us consider a curve as shown in the figure. Now, if (x, y) be any point on the curve, then the image of the point with respect to x-axis is $(x, -y)$. If the curve is symmetric, then both (x, y) and $(x, -y)$ will lie on the curve, i.e., both $f(x, y) = 0$ and $f(x, -y) = 0$, and therefore, we must have $f(x, -y) = \pm f(x, y)$.

Similarly, if by replacing x by $-x$ in the equation of the curve, the equation of curve remains unchanged, i.e., if $f(-x, y) = \pm f(x, y)$, then the curve is symmetrical about y-axis.

Note. Sometimes, we need to find whether a curve is symmetrical about the line $y = x$. If by interchanging x and y in the equation of the curve, the equation of curve remains unchanged, i.e., if $f(y, x) = f(x, y)$, then the curve is symmetrical about the line $y = x$.

Some examples:

Example 1. Consider the curve $x^2 + y^2 = a^2$, which is a circle with centre at $(0, 0)$ and radius a units. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis. This is shown in Fig. 2.

Example 2. Consider the curve $(x - a)^2 + y^2 = r^2$, which is a circle with centre at $(a, 0)$ (i.e., on x-axis) and radius r units. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis. This is shown in Fig. 3.

Example 3. Consider the curve $y^2 = 4ax$, which is a parabola with vertex at $(0, 0)$ and the x-axis being the axis. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. However, by replacing x by $-x$, the curve does not remain unchanged, so the curve is not symmetrical about the y-axis. This is shown in Fig. 4.

Example 4. Consider the curve $x^{2/3} + y^{2/3} = a^{2/3}$, an Astroid, shown in Fig. 4. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis.

(b) Rules to trace curves given in cartesian form

To trace curves in the cartesian form, say, $f(x, y) = 0$, we note the following points:

(i) **Symmetrical about the co-ordinate axes:** Check whether the given curve is symmetrical about the co-ordinate axes.

(ii) **Obtain the points of intersection with x-axis and y-axis:** To find the points of intersection of the curve with x-axis, we put $y = 0$ in the equation of the curve. Similarly, to find the points of intersection of the curve with y-axis, we put $x = 0$ in the equation of the curve.

(iii) **Obtain the boundary of the curve:** If by putting $x < -a$ or $x > b$ in the equation of the curve, y^2 becomes negative (i.e., y takes imaginary values), that means no part of the curve lies to the left of the line $x = -a$ or to the right of the line $x = b$. Similarly, if by putting $y < -c$ or $y > d$ in the equation of the curve, x^2 becomes negative (i.e., x takes imaginary values), that means no part of the curve lies below the line $y = -c$ or above the line $y = d$.

(iv) **Tangent of curve at origin:** If the curve passes through the origin and the equation of the curve is given by a polynomial equation, the equation of the tangent exists and is obtained by equating to zero, the lowest degree terms of the equation.

(v) **Asymptotes to the curve parallel to the co-ordinate axes:** Let the equation of the curve be given by a polynomial equation. Obtain the co-efficient of the highest degree term of x. If the co-efficient is a non-constant, i.e., a function of y, say, $\phi(y)$, then the asymptotes parallel to x-axis exists, and are given by $\phi(y) = 0$. Obtain the co-efficient of the highest degree term of y. If the co-efficient is a non-constant, i.e., a function of x, say, $\psi(x)$, then the asymptotes parallel to y-axis exists, and are given by $\psi(x) = 0$.

Tracing of curves in polar form

For a curve given in polar form $r = f(\theta)$ (or, $f(r,\theta) = 0$), if the equation of the curve remains unchanged by replacing θ by $-\theta$, then the curve is symmetrical about the initial line. If the curve is symmetrical about the initial line, obtain the values of the radius vector r for different values of θ in the range 0 to π, i.e., for the upper half; otherwise, obtain the values of the radius vector r for different values of θ in the range 0 to 2π.

Some standard curves

(a) $x^{2/3} + y^{2/3} = a^{2/3}$

This curve is known as 'Astroid'. The parametric equation of the curve may be written as $x = a\cos^3\theta, y = a\sin^3\theta$. The equation of given curve may be written as $\left(x^{1/3}\right)^2 + \left(y^{1/3}\right)^2 = a^{2/3}$. We see that replacing x by $-x$ and y by $-y$ does not change the equation of the curve, and therefore, the curve is symmetrical about both x and y axes. The curve meets the x-axis at the points $(a, 0)$ and $(-a, 0)$, and meets the y-axis at the points From the given equation, we can write $x = \left(a^{2/3} - y^{2/3}\right)^{3/2}$. It is clear that if $|y| > a$, then x becomes imaginary. Therefore, no part of the curve lies either above the line $y = a$ or below the line $y = -a$. Again, from the given equation, we can write $y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$. It is clear that if $|x| > a$, then y becomes imaginary. Therefore, no part of the curve lies either to the left of the line $x = -a$ or to the right of the line $x = a$. Thus, the complete curve can be drawn only if we can draw the portion of the curve lying in the first quadrant. For this, we compute the values of (x, y) corresponding to some values of θ in the range 0 to $\pi/2$ of the first quadrant, and present in the Table 1.

Table 1. Values of (x, y) corresponding to some values of θ

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
x	a	$\frac{3\sqrt{3}}{8}a$	$\frac{1}{2\sqrt{2}}a$	$\frac{1}{8}a$	0
y	0	$\frac{1}{8}a$	$\frac{1}{2\sqrt{2}}a$	$\frac{1}{2\sqrt{2}}a$	a

Thus, a rough sketch of the astroid is as shown in Fig. 1.

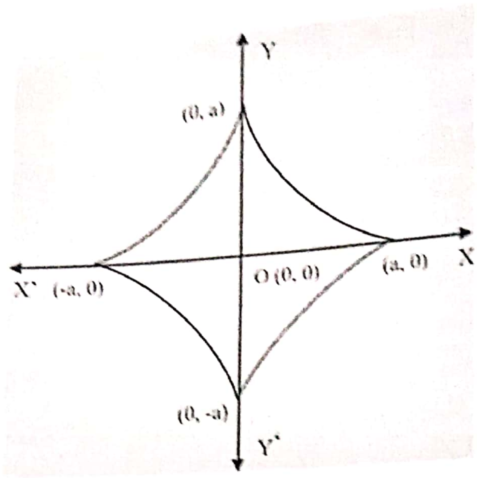


Fig. 1. A rough sketch of 'Astroid'

(b) $x^3 + y^3 = 3axy$

This curve is known as Folium of Descartes. This curve is symmetrical about the line $y = x$. The curve pass through the origin and meets the line $y = x$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$. The line $x + y + a = 0$ is the only asymptote of the curve (Ref. to Example of Chapter). A rough sketch of the curve is as shown in Fig. 2.

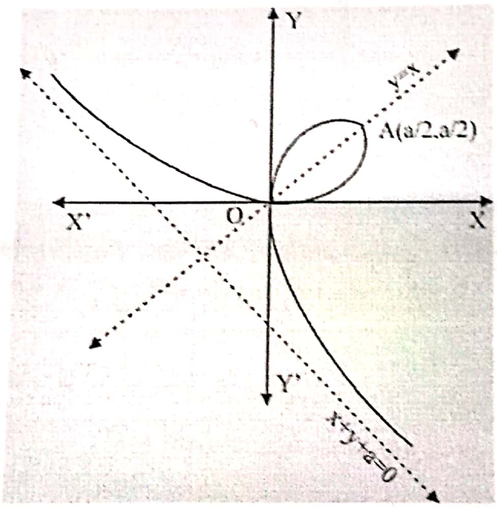


Fig. 2. A rough sketch of 'Folium of Descartes'

(c) $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$

This curve is known as Cycloid, and is traced by a fixed point taken on the rim of a cycle when the cycle is moved from one place to another on a plane surface without sliding. The sketch of the graph is as shown in Fig. 2. Here, θ varies from $-\pi$ to π.

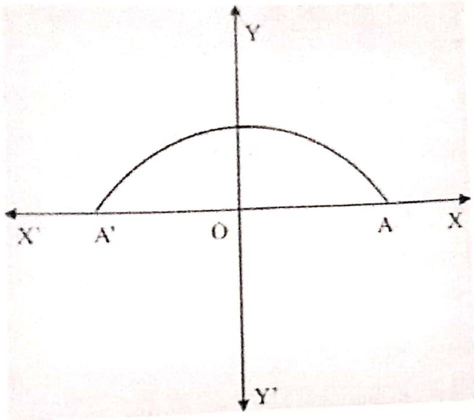


Fig. 3. A rough sketch of 'Folium of Descartes'

Note. Another form of cycloid:. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ Here, θ varies from 0 to 2π. The sketch of the graph is as shown in Fig. 3.

(d) $r = a(1 - \cos\theta)$

This curve is known as Cardioide. Since by replacing θ by $-\theta$, the equation of the curve remains unchanged, so the curve is symmetrical about the initial line. The values of radius vector r corresponding to some values of θ are given in Table 2. A rough sketch of the curve is as shown in Fig. 4.

Table 2. Length of radius vector r corresponding to some values of θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a\left(1-\frac{\sqrt{3}}{2}\right)$	$a\left(1-\frac{1}{\sqrt{2}}\right)$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

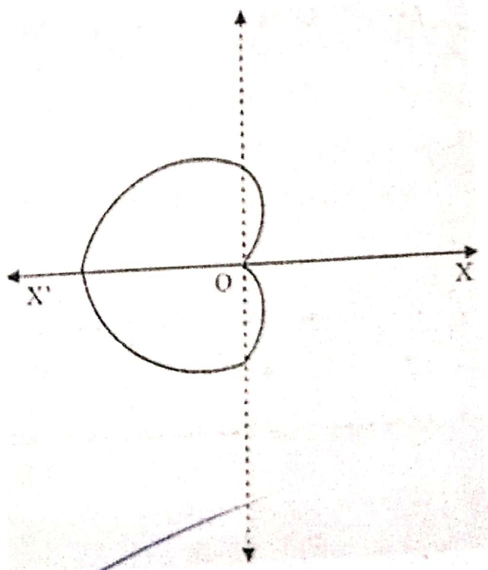


Fig. 4. A rough sketch of 'Cardioide'

Worked-out examples:

Example 1. Trace the curve represented by the equation $3ay^2 = x(x-a)^2$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis, but not symmetrical about the y-axis. The curve meets the x-axis at the points $(0, 0)$ and $(a, 0)$. The curve passes through the origin, and the tangent at origin is $x = 0$, i.e., y-axis. When $x < 0$, y^2 becomes negative, and hence, no part of the curve lies to the left of the line $x = 0$. But, as the value of x increases beyond a, the value of y also increases. Thus, a rough sketch of the curve is as shown in Fig. 5.

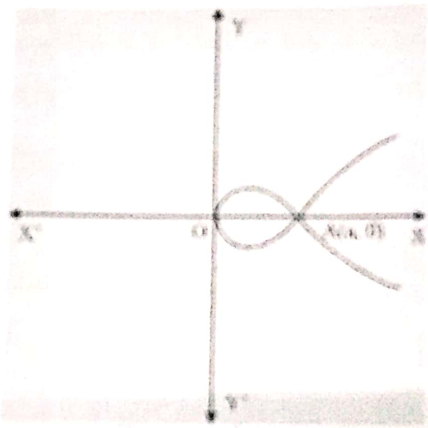


Fig. 5. A rough sketch of the curve given by the equation $3ay^2 = x(x-a)^2$

Example 2. Trace the curve represented by the equation $ay^2 = x^2(a-x)$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis. The curve meets the x-axis at the points $(0, 0)$ and $(a, 0)$. The curve passes through the origin, and the tangent at the origin is given by $ay^2 = ax^2$ (obtained by equating to zero, the lowest degree terms of the equation), i.e., $y = \pm x$. If we put some value of $x > a$, we see that y^2 becomes negative, which means that no part of the curve lies to the right of the line $x = a$. Thus, a rough sketch of the curve is as shown in Fig. 6.

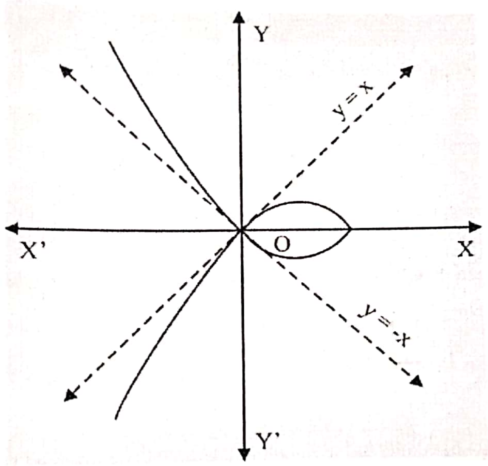


Fig. 6. A rough sketch of the curve given by the equation $ay^2 = x^2(a - x)$

Example 3. Trace the curve represented by the equation $y^2(2a - x) = x^3$, where $a > 0$ is a real constant.

Solution. The given curve is known as 'Cissoid of Diocles'. The curve is symmetrical about the x-axis. The curve meets the x-axis only at the origin. The curve passes through the origin, and the tangent at the origin is given by $y = 0$, i.e., x-axis. We see that when $x < 0$ or $x > 2a$, y^2 takes negative value, which means no part of the curve lies to the left of the line $x = 0$ or to the right of the line $x = 2a$. Moreover, the straight line $x = 2a$ is an asymptote of the given curve. Thus, a rough sketch of the curve is as shown in Fig. 7.

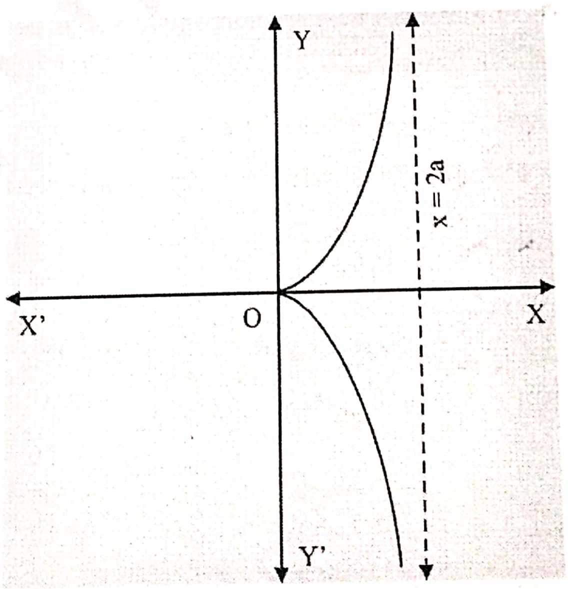


Fig. 7. A rough sketch of the curve given by the equation $ay^2 = x^2(a - x)$

Example 4. Trace the curve represented by the equation $x^{1/2} + y^{1/2} = a^{1/2}$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the line $y = x$. The curve meets the x-axis at the point $(a, 0)$ and meets the y-axis at $(0, a)$. From the given equation, we see that neither x nor y can take negative value, and hence, the curve lies in first quadrant only. Thus, a rough sketch of the curve is as shown in Fig. 8.

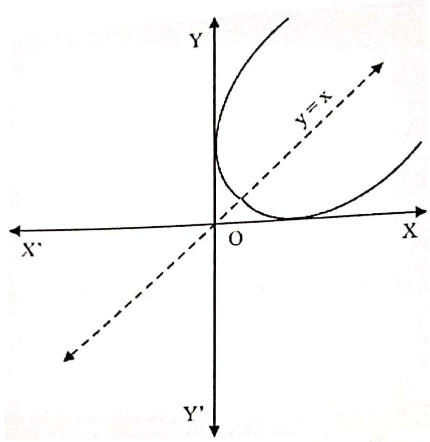


Fig. 8. A rough sketch of the curve given by the equation $x^{1/2} + y^{1/2} = a^{1/2}$

Example 5. Trace the curve represented by the equation $xy^2 + (x + a)^2(x + 2a) = 0$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis, but not symmetrical about the y-axis. The curve meets the x-axis at the points $(-2a, 0)$ and $(-a, 0)$. The given equation may be re-written as $y^2 = -\frac{(x+a)^2(x+2a)}{x}$, which shows that if $x < -2a$ or $x > 0$, then y^2 becomes negative. Thus, no part of the curve lies to the left of the line Moreover, $x = 0$, i.e., y-axis is an asymptote to the curve. Considering all these points, a rough sketch of the curve is as shown in Fig. 9.

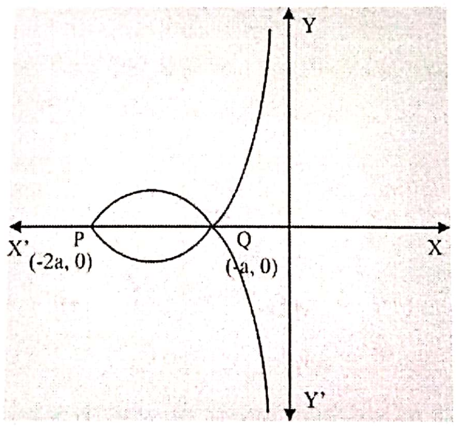


Fig. 9. A rough sketch of the curve given by the equation $xy^2 + (x + a)^2(x + 2a) = 0$

Example 6. Trace the curve represented by the equation $a^2y^2 = x^2(a^2 - x^2)$, where $a > 0$ is a real constant.

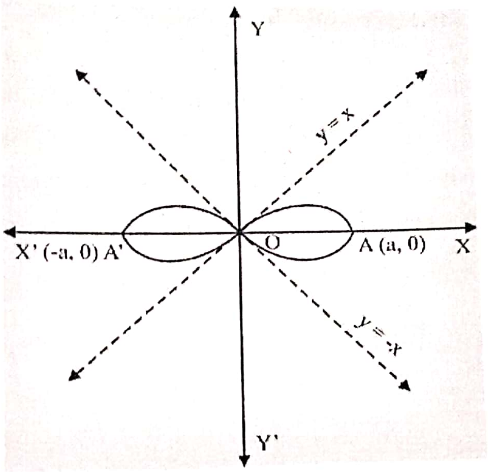


Fig. 10. A rough sketch of the curve given by the equation $a^2y^2 = x^2(a^2 - x^2)$

Solution. The curve is symmetrical about both the co-ordinate axes. The curve meets x-axis at the points $(-a, 0)$ and $(a, 0)$. The curve passes through the origin and $y = \pm x$ are the tangents at the origin. If $x > a$ or $x < -a$, then y^2 becomes negative, which means that no part of the curve lies to the left of the line $x = -a$ and to the right of the line $x = a$. Considering all these points, a rough sketch of the curve is as shown in Fig. 10.

Note. Bernoulli's Lemniscate Page no 23

Example 7. Trace the curve represented by the equation $y = \frac{a^3}{a^2+x^2}$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the y-axis only. The curve meets y-axis at the point $(0, a)$. The equation of the curve may be re-written as $x^2 = \frac{a^2(a-y)}{y}$, which shows that if $y > a$, then x^2 takes negative value. Thus, no part of the curve lies above the line $y = a$. Moreover, $y = 0$ is an asymptote to the curve. Considering all these points, a rough sketch of the curve is as shown in Fig. 11.

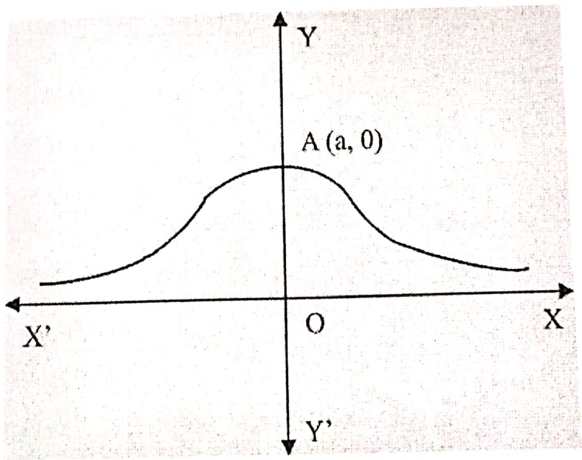


Fig. 11. A rough sketch of the curve given by the equation $y = \frac{a^3}{a^2+x^2}$

Example 8. Trace the curve represented by the equation $x^2y^2 = a^2(y^2 - x^2)$, where $a > 0$ is a real constant.

Solution. The given curve is symmetrical about both the co-ordinate axes. The curve meets x-axis only at the origin. The curve passes through the origin and $y = \pm x$ are the tangents at the origin. If $x > a$ or $x < -a$, then y^2 becomes negative, which means that no part of the curve lies to the left of the line $x = -a$ and to the right of the line $x = a$. Considering all these points, a rough sketch of the curve is as shown in Fig. 12.

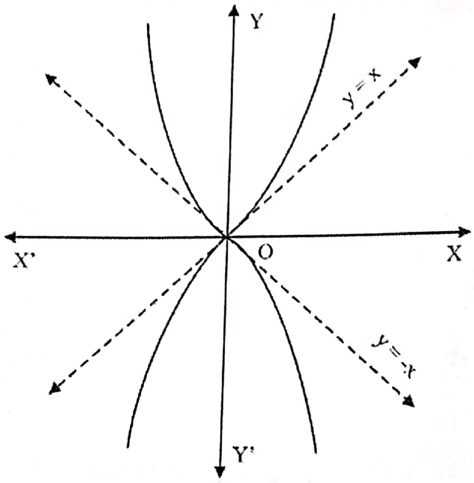


Fig. 12. A rough sketch of the curve given by the equation $x^2y^2 = a^2(y^2 - x^2)$

Example 9. Trace the curve represented by the equation $x^4+y^4 = 2a^2xy$, where $a > 0$ is a real constant.

Solution. The given curve is symmetrical about the straight line $y = x$, and meets the line $y = x$ at the points (a, a) and $(-a, -a)$. The curve passes through the origin, and the tangents at the origin are $x = 0$ and $y = 0$. Considering all these points, a rough sketch of the curve is as shown in Fig. 13.

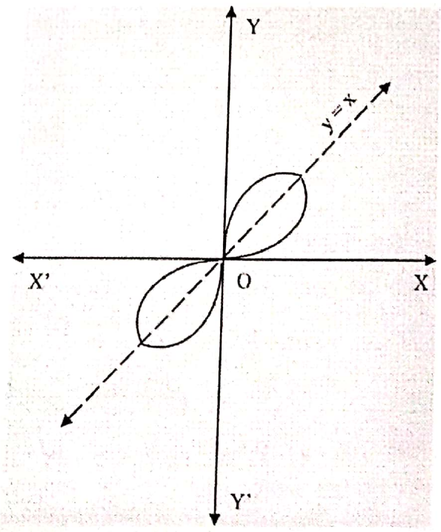


Fig. 12. A rough sketch of the curve given by the equation $x^4+y^4 = 2a^2xy$

Exercises.

Trace the curves for the following equations.

1. $y^2(a - x) = x(x - b)^2$
2. $y^2(a + x) = x^2(b - x)$
3. $y^2 = (x - a)^3$
4. $9y^2 = (x + 7)(x + 4)^2$
5. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ (This curve is known as Astroid or Four Cusped Hypocycloid)

References.

1. Book: Application of Calculus: Theory and Problems by Sitansu Bandhopadhyay and Sunil Kumar Maity
2. Website: https://math.buet.ac.bd/public/faculty_profile/files/766768001.pdf

Journal of Computational and Applied Mathematics

http://dx.doi.org/10.1016/j.cam.2015.09.004

LOGARITHMIC SPIRALS AND CONTINUE TRIANGLES

GIUSEPPINA ANATRIELLO[A] AND GIOVANNI VINCENZI[B,1]

[A] Department of Architecture, University of Naples "Federico II", Via Monteoliveto, 3 I-80134 Napoli, Italy

[B] Department of Mathematic, University of Salerno, Via Giovanni Paolo II, 132 I-84084 Fisciano (SA), Italy

ABSTRACT. In this article we will use some special triangles, to construct polygonal chains that describe the families of logarithmic spirals, among which the celebrated Golden Spiral, Spira solaris and Pheidia Spiral.

Key words: Kepler triangles, Almost congruent triangles, Continue triangles, Discretized spirals, Golden Spirals, Logarithmic spirals, Logarithmic elliptic spirals, Golden mean.

1. INTRODUCTION

A *logarithmic spiral* is a plain curve whose equation in polar coordinate (ρ, θ) is $\rho = te^{(h\theta)}$. The term h is a positive number called the *growth constant* of the spiral, and t is a *constant of the spiral* depending on the choice of the initial condition $\theta = 0$. Note that θ increases anti-clockwise. The Cartesian representation of a logarithmic spiral is

$$(1.1) \qquad \begin{cases} x(\theta) = \rho(\theta)cos(\theta) = te^{(h\theta)}cos(\theta) \\ y(\theta) = \rho(\theta)sin(\theta) = te^{(h\theta)}sen(\theta), \end{cases}$$

thus the distance from the origin (Pole) of $(x(\theta), y(\theta))$ increases exponentially when θ increases (anti-clockwise). Sometimes this kind of spiral is more precisely called a *Left hand logarithmic spiral*, to distinguish it from a *Right hand logarithmic spiral*, whose equation is of the type $\rho = te^{(-h\theta)}$. For the latter type of spirals the distance from the Pole of $(x(\theta), y(\theta))$ decreases exponentially when θ increases.

The most celebrated curve of this type is certainly the *Golden Spiral*, which is a logarithmic spiral whose growth constant is $(2/\pi)\lg(\Phi)$, where Φ is the "Golden Mean".

Looking at the equation of the Golden Spiral

$$\rho = e^{(2/\pi)\lg(\Phi)\theta} \quad \text{with } \textit{starting point (1,0)},$$

we note that for $\theta = 0$ we have $\rho = 1$, and for $\theta = \pi/2$ we have $\rho = \Phi$. More generally it can be easily seen that, a golden spiral gets wider (or further from its origin) by a factor of Φ for every quarter turn it makes; therefore " Φ^4 " gives a measurement of the *growth factor* of this spiral after a complete turn around the pole.

The Golden Spiral was first described by Descartes, and then studied by the Swiss mathematician Jakob Bernoulli (1654-1705), who called it *Spira mirabilis*, and dedicated to it the famous motto " *Eadem Mutata resurgo* " which is inscribed on his tombstone.

Approximations of logarithmic spirals can occur in nature (for example, the arms of spiral galaxies or phyllotaxis of leaves); golden spirals are one special case of these. It is sometimes stated that spiral galaxies and nautilus shells get wider in the pattern of a golden spiral, and hence are related to both Φ and the Fibonacci series. In truth, spiral galaxies and nautilus shells (and many mollusk shells) exhibit logarithmic spiral growth, but at a variety of angles usually distinctly different from that of the golden spiral. This

[1]Corresponding author: tel/fax. +3989968225, vincenzi@unisa.it

pattern allows the organism to grow without changing shape (see [3] and [7]).
Also, we note a very interesting connection between the DNA-spiral and some Fibonacci-like sequences (see [15]). This highlights the relevant connection between recursive sequences and spirals (see also [8, 1, 19] and the references therein).

The German Mathematician Johannes Kepler (1571-1630) was the first to study the nature of the logarithmic spirals, and its possible *discretezations.* He was also attracted by their shape and their applications in astronomy. For this purpose he considered a logarithmic spiral, with a lower factor growth (Φ^2): The *Spira Solaris,* whose equation is given by:

$$\rho = e^{(1/\pi)\lg(\Phi)\theta} = \Phi^{\theta/\pi} \quad \text{with starting point (1,0).}$$

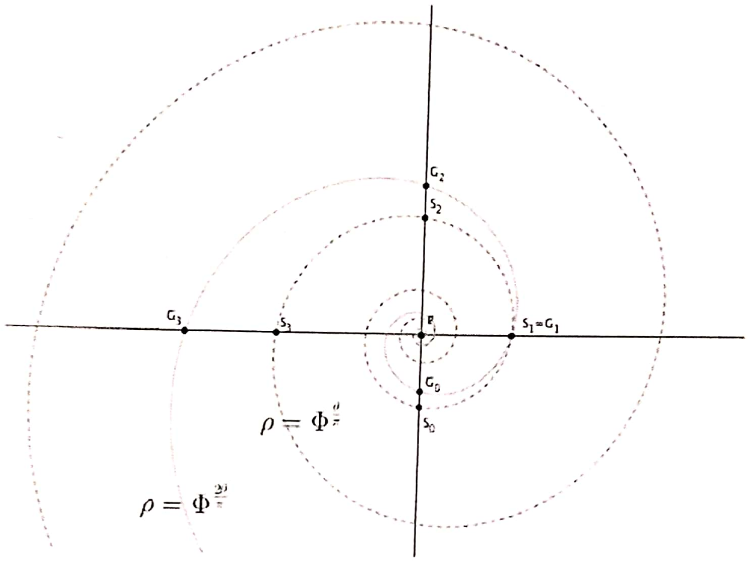


FIGURE 1.1. Dotted Red line is the Spira Solaris, Gold line is the Golden Spiral

Another celebrated logarithmic spiral is the *Pheidia Spiral,* whose equation is:

$$\rho = e^{(1/2\pi)\lg(\Phi)\theta} = \Phi^{\theta/2\pi}, \quad \text{with starting Point (1,0).}$$

Note that Pheidia Spirals, Spira Solaris, and Golden Spiral have respectively the following "growth": Φ, Φ^2, Φ^4.

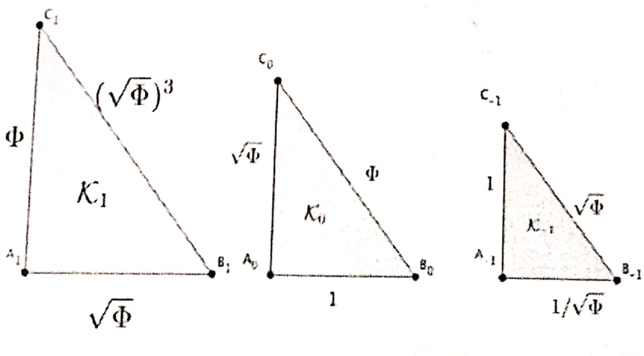


FIGURE 1.2. Examples of Kepler triangles

Connected to the Spira Solaris, there are the *Golden right triangle* or *Kepler triangle,* that is any right triangle $\mathcal{K}$ the lengths of whose sides, $a > b > c$ satisfy the proportion:

$a : b = b : c$. The Figure 1.2 shows examples of Kepler triangles. For every integer n, we will denote by $\mathcal{K}_n$ a triangle the measurement of whose sides are $(\sqrt{\Phi})^n$, $(\sqrt{\Phi})^{n+1}$ and $(\sqrt{\Phi})^{n+2}$ respectively (Φ is the golden mean). It is easy to check that each $\mathcal{K}_n$ is a Kepler triangle.

Following Pennisi (see[14]), we will say that a triangle (not necessarily right) is *continue* if the lengths of whose sides, $a > b > c$ satisfy the proportion: $a : b = b : c$. In [14] Pennisi studied the connection between Kepler triangles and spirals, and highlighted that (see Figure 1.3)

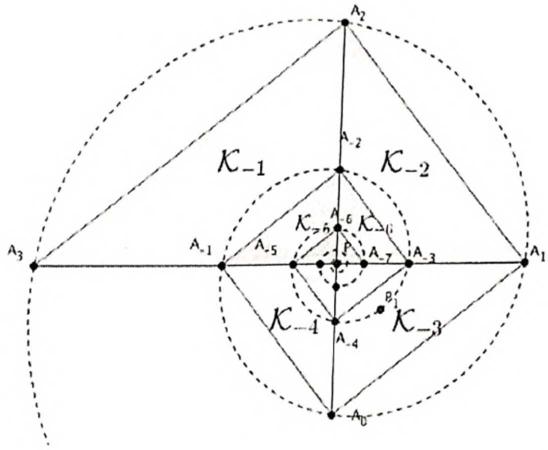


FIGURE 1.3. $\mathcal{K}_{n-1} \cup \mathcal{K}_{n-2}$, is congruent to $\mathcal{K}_n$

the polygonal chain

$$\text{“} \ldots A_{-4}, A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3, \ldots \text{”}$$

can describe (or *discretize*) the Spira Solaris (in his paper he called it *continue spiral*). In the figure it is represented by a dotted line.

There is ample literature from Descartes to the present, that also shows how some polygonal chains can describe other celebrated logarithmic curves, and their impact on sciences such as Astronomy, Physics, Biology, Botany, Medicine (see [2, 7, 5, 12] and reference therein). The applications of logarithmic spirals are very relevant also in Architecture and Engineering.

It arises the natural question: Are there other logarithmic spirals that can be described by using continue triangles?

In this article we will give some positive answers. We will investigate the properties of polygonal chains constructed by continue triangles. We will use a new computational techniques (see Lemma 3.1 and Corollary 3.2), in order to arrive at a description of the family of logarithmic spirals (see Theorem 4.1 and section 5). In section 2 we recall the main properties of a continue triangle (not necessarily right). In section 3 we show the geometric algorithm for the construction of some polygonal chains: the (r, k)-*male spirals*. In section 4 we will see how some families of logarithmic spirals, among which Pheidia Spiral, Spira Solaris, and Golden Spiral, can be described by these "special" polygonal chains. The paper is as self-contained as possible, and has been written providing a soft approach to the study of logarithmic spirals and encouraging the use of professional mathematical software.

2. Chains of continue triangles

Consider a triangle $\mathcal{T}$ whose vertices are A, B and C, and define a, b, c as the measurement of the sides that are opposite respectively to A, B and C. In the following we will denote triangle $\mathcal{T}$ by (A, B, C) or (a, b, c).
Clearly a triangle is continue, if the measurement of its sides are three consecutive numbers of a geometric progression, but the converse is not true: if we consider three numbers in geometric progression, we cannot say that they are the length of the sides of a continue triangle. To this end, it is enough to consider the numbers $2, 4, 8$: they do not satisfy the triangular property. Moreover, if $a > b > c$ and (a, b, c) is a continue triangle, then also (b, c, d) is a continue triangle, having chosen d such that $b : c = c : d$. Clearly the triangles (a, b, c) and (b, c, d) are similar and have two sides congruent. Following R. T. Jones and B. B. Peterson we will call such pairs of triangles *almost congruent* (see [13]).

By definition and triangular inequality we have two elementary properties of continue triangles (see also [13] Theorem 1, and [14] page 22).

Theorem 2.1. *Let $\mathcal{T} = (a, b, c)$ be a triangle. If $\mathcal{T}$ is a continue triangle, then a, b, c are in geometric progression of mean lying in $(1/\Phi, \Phi) \setminus \{1\}$. Conversely for every geometric progression kr, kr^2, kr^3, where k is a positive real number and the mean r lies in $(1/\Phi, \Phi) \setminus \{1\}$, the triangle (kr, kr^2, kr^3) is continue.*

Remark 2.1. In the above statement it is easy to see that if $r \in (1/\Phi, 1/\sqrt{\Phi}) \cup (\sqrt{\Phi}, \Phi)$ the corresponding triangles are obtuse; and if $r \in (1/\sqrt{\Phi}, \sqrt{\Phi}) \setminus \{1\}$ the corresponding triangles are acute. Moreover, if r is exactly $\sqrt{\Phi}$ we have that (kr, kr^2, kr^3) define a Kepler triangle, for every positive k.

The following figure shows examples of pairs of almost congruent triangles, with integer sides. (see [4] and [13] p. 182):

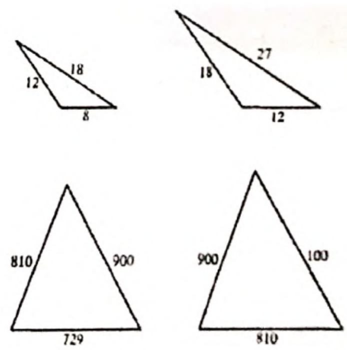


FIGURE 2.1. Pairs of almost congruent triangles with integer sides. Each of them is a continue triangle

Let $r \in (1/\Phi, \Phi)$, and for each integer n consider the continue triangle $\mathcal{T}_n = (r^n, r^{n+1}, r^{n+2})$. In the following, by writing "$\mathcal{T}_n < \mathcal{T}_m$" we mean that the area of $\mathcal{T}_n$ is less than the area of $\mathcal{T}_m$.

Clearly, for every $r \in (1/\Phi, \Phi)$ the sets $\{\mathcal{T}_n\}_{n\in\mathbb{Z}} = \{(r^n, r^{n+1}, r^{n+2})\}_{n\in\mathbb{Z}}$ and $\{\mathcal{T}'_n\}_{n\in\mathbb{Z}} = \{((\frac{1}{r})^n, (\frac{1}{r})^{n+1}, (\frac{1}{r})^{n+2})\}_{n\in\mathbb{Z}}$ coincide.
These two chains of continue triangles exhibit opposite behavior:
If $1 < r$, then the chain $(\mathcal{T}_n)$ is ascendant, while $(\mathcal{T}'_n)$ is descendant;
if $r < 1$, then the chain $(\mathcal{T}_n)$ is descendant and $(\mathcal{T}'_n)$ is ascendant.

Thus we may restrict our considerations to the case of $r \in (1, \Phi)$. In particular we have:

$$\mathcal{T}_n < \mathcal{T}_m \quad \text{and} \quad \mathcal{T}'_n > \mathcal{T}'_m. \quad \forall n < m \in \mathbb{Z}.$$

3. The class of (r, k)-male spirals

Let $r \in (1, \Phi)$, and consider the chain $\{\mathcal{T}_n\}_{n\in\mathbb{Z}}$ as defined in section 2. In the following, for every integer n we set $A_{n+1}, A_{n+2}, A_{n+3}$, as the vertices of $\mathcal{T}_n$. Thus

$$\mathcal{T}_n = (r^n, r^{n+1}, r^{n+2}) = (A_{n+1}, A_{n+2}, A_{n+3}).$$

For every positive integer k we can consider the subchain $\{\mathcal{T}_{nk}\}_{n\in\mathbb{Z}}$ and make the following construction:

Starting from $\mathcal{T}_0 = (1, r^1, r^2) = (A_1, A_2, A_3)$, for every integer k we can consider the triangles $\mathcal{T}_k = (r^k, r^{k+1}, r^{k+2}) = (A_3, A_4, A_5)$ and $\mathcal{T}_{-k} = (r^{-k}, r^{-k+1}, r^{-k+2}) = (A_1, A_0, A_{-1})$, and by using their similarity we can draw them as follows:

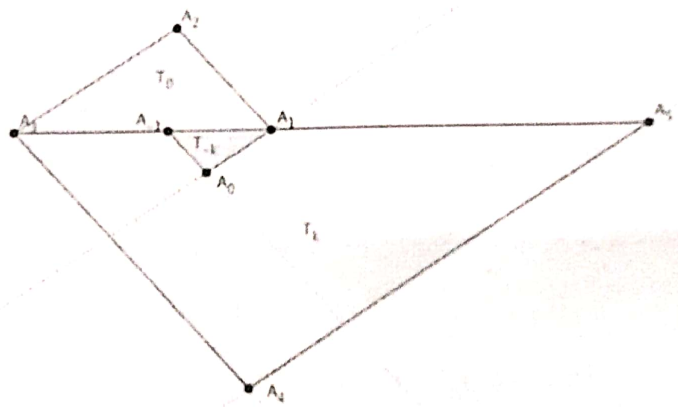


FIGURE 3.1. The construction of the (r, k)-male spiral

Now we can join triangle $\mathcal{T}_{-k} = (A_1, A_0, A_{-1})$, with the smaller

$$\mathcal{T}_{-2k} = (r^{-2k}, r^{-2k+1}, r^{-2k+2}) = (A_{-1}, A_{-2}, A_{-3});$$

and triangle $\mathcal{T}_k = (A_3, A_4, A_5)$ with the larger one

$$\mathcal{T}_{2k} = (r^{2k}, r^{2k+1}, r^{2k+2}) = (A_5, A_6, A_7).$$

The result of this iterated process is a polygonal chain that we call (r, k)-*male spiral*. We will denote it by $\mathcal{P}_{r,k}$. For this "spiral" we have the following result:

Lemma 3.1. *Let $r \in (1, \Phi)$ and k be a positive integer, and consider the (r, k)-male spiral $\mathcal{P}_{r,k}$. Then all the vertices A_{2n}, indexed by an even number, lie on the same line s. In particular, if we define P as the intersection between s and the line containing all the vertices A_{2n+1}, then each A_{1-4n} lies to the right of P and each A_{3-4n} lies to the left of P, for every integer n. Moreover: the following relations hold:*

$$\text{(a) } |\overline{A_3P}| = \sum_{n=0}^{\infty} |\overline{A_{3-4n}A_{3-4(n+1)}}| = |\overline{A_3A_{-1}}| + |\overline{A_{-1}A_{-5}}| + \ldots = \frac{r^{2+k}}{r^k + 1};$$

(b) $|\overline{A_1P}| = \sum_{n=0}^{\infty} |\overline{A_{1-4n}A_{1-4(n+1)}}| = |\overline{A_1A_{-3}}| + |\overline{A_{-3}A_{-7}}| + \ldots = \dfrac{r^2}{r^k+1}$;

(c) $|\overline{A_2P}| = \sum_{n=0}^{\infty} |\overline{A_{2-4n}A_{2-4(n+1)}}| = |\overline{A_2A_{-2}}| + \ldots = \dfrac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$;

(d) $|\overline{A_0P}| = \sum_{n=0}^{\infty} |\overline{A_{4n}A_{-4(n+1)}}| = |\overline{A_0A_{-4}}| + \ldots = \dfrac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{r^k(r^k+1)}$.

Proof. We start by showing that A_2, A_0 and A_4 are collinear. To this end, consider the line a through the vertices A_1 and A_0, and define F as the intersection of a with $\overline{A_3A_4}$.

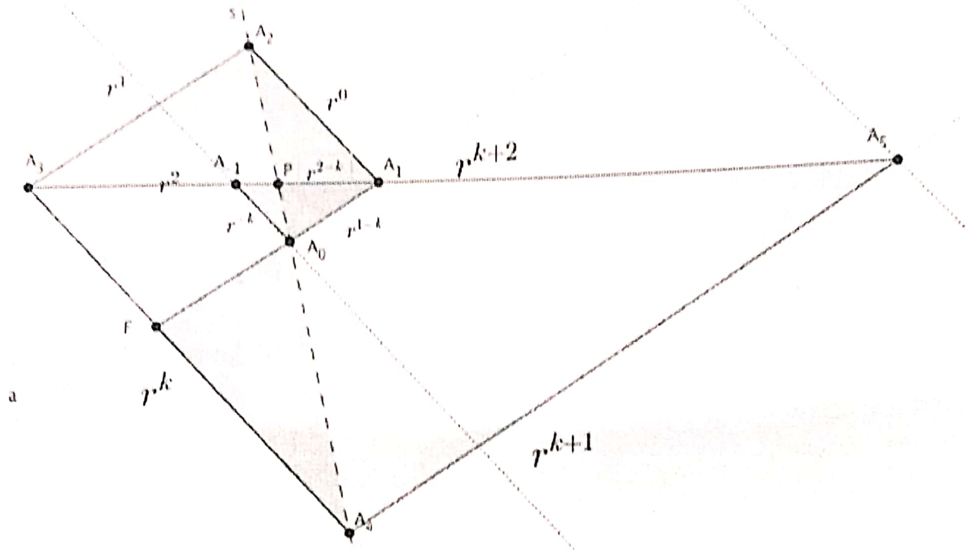


FIGURE 3.2. The Pole of the (r,k)-male spiral

The angles $A_2\hat{A}_1A_0$ and $A_0\hat{F}A_4$ are congruent; moreover, the following proportion holds: $\overline{A_2A_1} : \overline{A_1A_0} = \overline{FA_4} : \overline{FA_0}$, in fact, the measurements of these segments result as:

$$1 : r^{1-k} = r^k - 1 : r - r^{1-k},$$

so that the triangles (A_2, A_1, A_0) and (A_0, F, A_4) are similar. In particular, $A_1\hat{A}_0A_2$ and $F\hat{A}_0A_4$ are congruent. We have proved that three vertices A_0, A_2, A_4, of the three consecutive continue triangles $\mathcal{T}_{-k}, \mathcal{T}_0$ and $\mathcal{T}_k$ are collinear. Iterating the process "up and down" we find that all A_{2n} are collinear.

In particular, for each integer n, the vertex A_{1-4n} lies to the right of the line s, and A_{3-4n} lies to the left.

Now we note that P is the intersection of all $\mathcal{T}_k$, so that:

$$\bigcup_{n\in\mathbb{N}_0} \overline{A_{3-4n}A_{3-4(n+1)}} = \overline{A_3P}; \quad \bigcup_{n\in\mathbb{N}_0} \overline{A_{1-4n}A_{1-4(n+1)}} = \overline{A_1P};$$

$$\bigcup_{n\in\mathbb{N}_0} \overline{A_{4-4n}A_{4-4(n+1)}} = \overline{A_4P}; \quad \bigcup_{n\in\mathbb{N}_0} \overline{A_{2-4n}A_{2-4(n+1)}} = \overline{A_2P}.$$

It follows that:

(a). $|\overline{A_3A_{-1}}| = r^2 - r^{2-k}$, $|\overline{A_{-1}A_{-5}}| = r^{2-2k} - r^{2-3k}$ and in general for every integer $n \geq 0$ $|\overline{A_{3-4n}A_{3-4(n+1)}}| = r^{2-2nk} - r^{2-(2n+1)k}$. Therefore

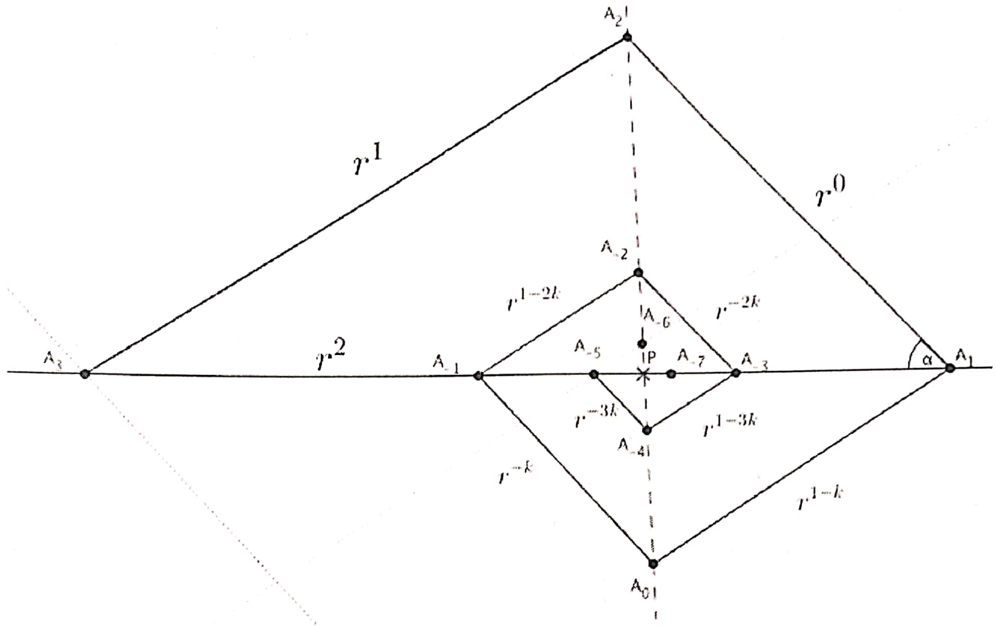


FIGURE 3.3. The growth of the (r,k)-male spiral

$$\overline{A_3P} = \sum_{n=0}^{\infty} |\overline{A_{3-4n}A_{3-4(n+1)}}| = \sum_{n=0}^{\infty} r^{2-2nk} - r^{2-(2n+1)k} = \sum_{n=0}^{\infty} r^{2-2nk}(1-r^{-k})$$

$$= r^2(1-r^{-k})\sum_{n=0}^{\infty} r^{-2nk} = r^2(1-r^{-k})\sum_{n=0}^{\infty}\left(\frac{1}{r^{2k}}\right)^n = r^2(1-r^{-k})\frac{1}{1-1/r^{2k}}$$

$$= \frac{r^{2+k}(r^k-1)}{r^{2k}-1} = \frac{r^{2+k}}{r^k+1}$$

(b) It follows from (a) that: $\overline{A_1P} = \overline{A_1A_3} - \overline{A_3P} = r^2 - \frac{r^{2+k}}{r^k+1} = \frac{r^2}{r^k+1}$.

(c) To verify this item, we will make double use of Carnot' theorem. Following the figure 3.3, we set $\alpha = A_2\hat{A}_1A_3$. Then

i) $|\overline{A_3A_2}|^2 = |\overline{A_3A_1}|^2 + |\overline{A_2A_1}|^2 - 2|\overline{A_3A_1}||\overline{A_2A_1}|cos(\alpha)$, so we have

$$cos(\alpha) = \frac{|\overline{A_3A_1}|^2 + |\overline{A_2A_1}|^2 - |\overline{A_3A_2}|^2}{2|\overline{A_3A_1}||\overline{A_2A_1}|} = \frac{r^4 + r^0 - r^2}{2r^2r^0} = \frac{r^4+1-r^2}{2r^2}.$$

ii) $|\overline{PA_2}|^2 = |\overline{PA_1}|^2 + |\overline{A_2A_1}|^2 - 2|\overline{PA_1}||\overline{A_2A_1}|cos(\alpha)$

$$= \left(\frac{r^2}{r^k+1}\right)^2 + 1 - 2\frac{r^2}{r^k+1}r^0\frac{r^4+1-r^2}{2r^2}$$

$$= \frac{r^4}{(r^k+1)^2} + 1 - \frac{r^4+1-r^2}{r^k+1} = \frac{r^4}{(r^k+1)^2} + \frac{r^{2k}+2r^k+1}{(r^k+1)^2} - \frac{(r^k+1)(r^4+1-r^2)}{(r^k+1)^2}$$

$$= \frac{r^4+r^{2k}+2r^k+1-r^{k+4}-r^k+r^{k+2}-r^4-1+r^2}{(r^k+1)^2} = \frac{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}{(r^k+1)^2}$$

Therefore

$$|\overline{PA_2}| = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}.$$

(d) The triangles (A_{-1}, P, A_0) and (A_1, P, A_2) are similar so that:

$$\overline{A_2A_1} : \overline{A_{-1}A_0} = \overline{A_2P} : \overline{PA_0} \text{ and } r^0 : r^{-k} = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)} : \overline{PA_0}.$$

Therefore $\overline{PA_0} = \frac{r^{-k}\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$, and the last relation d is verified.

□

In the next we will refer to the point P of the statement above as *the pole* of the male spiral.

Example 3.1. As a particular male spiral we may consider the $(\sqrt{\Phi}, 2)$-male spiral. Using (a) and (c) of Lemma 3.1 we have:

$$|\overline{A_1P}| = \frac{r^2}{r^k+1} = \frac{\sqrt{\Phi}^2}{\sqrt{\Phi}^2+1} = \frac{\Phi}{\Phi+1} = \frac{1}{\Phi};$$

$$|\overline{A_2P}| = 1/\sqrt{\Phi} = \frac{\sqrt{\Phi^2+\Phi-\Phi^3+\Phi^2+\Phi}}{(\Phi+1)} = \frac{\sqrt{2(\Phi^2+\Phi)-\Phi(\Phi+1)}}{(\Phi+1)}$$

$$= \sqrt{\frac{\Phi(\Phi+1)}{(\Phi+1)^2}} = \sqrt{\frac{\Phi}{\Phi+1}} = 1/\sqrt{\Phi}.$$

Thus (A_1, P, A_2) gives Kepler triangle $\mathcal{K}_{-2}$ (see also Figure 1.3). Here the pole P is exactly the intersection of the height of (A_1, A_2, A_3) with its hypotenuse.

Corollary 3.2. *Let $r \in (1, \Phi)$ and k be a positive integer, and consider the (r, k)-male spiral, $\mathcal{P}_{r,k}$, as in the lemma above (see figure 3.3). Then the distance of any A_m from the pole P of $\mathcal{P}_{r,k}$ increases by a factor of r^{2k} for each (positive) turn around the pole.*

Proof. We will examine four cases.

(a) A_m is to the left of P, that is $m \equiv 3(mod4)$, or in other word $m = 3-4n$ for some $n \in \mathbb{Z}$. Starting from $\overline{A_3P}$, we have: $|\overline{A_{-1}P}| = |\overline{A_3P}| - |\overline{A_3A_{-1}}| = \frac{r^{2+k}}{r^k+1} - r^2 + r^{2-k} = \frac{r^{2-k}}{r^k+1}$, and hence

$$\frac{|\overline{A_3P}|}{|\overline{A_{-1}P}|} = \frac{r^{2+k}}{r^k+1}\frac{r^k+1}{r^{2-k}} = r^{2k}. \tag{3.1}$$

On the other hand, by construction (see also item (a) of Lemma 3.1), $\frac{|\overline{A_7A_3}|}{|\overline{A_3A_{-1}}|} = r^{2k}$. Using the elementary properties of the proportions, we have: $\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7A_3} : \overline{A_3A_{-1}}$, thus

$\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7A_3} + \overline{A_3P} : \overline{A_3A_{-1}} + \overline{A_{-1}P}$, that is $\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7P} : \overline{A_3P}$. In particular

(3.2)
$$\frac{|\overline{A_7P}|}{|\overline{A_3P}|} = r^{2k}.$$

Now by iterating this process, we see that we can extend the equations 3.1 and 3.2 to any $n \equiv 3(mod4)$:

(3.3)
$$\frac{|\overline{A_{3-4n}P}|}{|\overline{A_{3-4(n+1)}P}|} = r^{2k}.$$

(b) A_m is on the right of P, that is $m \equiv 1(mod4)$. Note that:

(3.4)
$$|\overline{A_{1-4n}A_{1-4(n+1)}}| = r^{2-(2n+1)k} - r^{2-(2n+2)k} = r^{2-(2n+1)k}(1 - r^{-k})$$
$$= r^{2-k}(1 - r^{-k})r^{-(2n)k} = r^{2-k}(1 - r^{-k})(\frac{1}{r^{2k}})^n$$

Starting from $\overline{A_1P}$, by Lemma 3.1 (b), we have:
$|\overline{A_{-3}P}| = |\overline{A_1P}| - |\overline{A_1A_{-3}}| = \frac{r^2}{r^k+1} - r^{2-k}(1-r^{-k}) = \frac{r^2-r^2+r^{2-k}-r^{2-k}+r^{2-2k}}{r^k+1} = \frac{r^{2-2k}}{r^k+1}$.
It follows that

(3.5)
$$\frac{|\overline{A_1P}|}{|\overline{A_{-3}P}|} = \frac{r^2}{r^k+1}\frac{r^k+1}{r^2r^{-2k}} = r^{2k}.$$

On the other hand, by equation 3.4 $\frac{|\overline{A_5A_1}|}{|\overline{A_1A_{-3}}|} = \frac{r^{2-k}(1-r^{-k})(\frac{1}{r^{2k}})^{-1}}{r^{2-k}(1-r^{-k})(\frac{1}{r^{2k}})^0} = r^{2k}$.
Proceeding as in the second part of the proof of item (a), we obtain

(3.6)
$$\frac{|\overline{A_5P}|}{|\overline{A_1P}|} = r^{2k}, \text{ and more generally } \frac{|\overline{A_{1-4n}P}|}{|\overline{A_{1-4(n+1)}P}|} = r^{2k}.$$

(c) A_m is above P, that is $m \equiv 2(mod4)$. Set $m = 2 - 4n$. Note that the triangles $(A_{2-4n}PA_{3-4n})$ and $(A_{2-4(n+1)}PA_{3-4(n+1)})$ are similar. Then:
$$\frac{|\overline{A_{3-4n}P}|}{|\overline{A_{3-4(n+1)}P}|} = \frac{|\overline{A_{2-4n}P}|}{|\overline{A_{2-4(n+1)}P}|},$$
and by (a)
$$\frac{|\overline{A_{2-4n}P}|}{|\overline{A_{2-4(n+1)}P}|} = r^{2k}.$$

(d) A_m is below P, that is $m \equiv (mod4)$. In this case we use the similarity of the triangles $(A_{1-4n}PA_{-4n})$ and $(A_{-3-4n}PA_{-4(n+1)})$ and the item (b).

□

Remark 3.1. Clearly we may consider the construction of a male spiral by starting from any triangle; but in the general case the above lemma does not hold, as simple considerations show (see figure 3.4). For example, starting from the triangle $\mathcal{T}_0 = (A_1, A_2, A_3)$ where $\overline{A_1A_2} = 3$, $\overline{A_2A_3} = 4$ and $\overline{A_1A_3} = 6$, we can consider the similar smaller triangle $\mathcal{T}_{-1}$ the lengths of whose sides are $3/2, 2, 3$, and the larger one $\mathcal{T}_1$ the lengths of whose sides are $6, 8, 12$. If we make the same construction as in figure 3.2, we find that the vertices A_0, A_2, A_4 are not collinear.

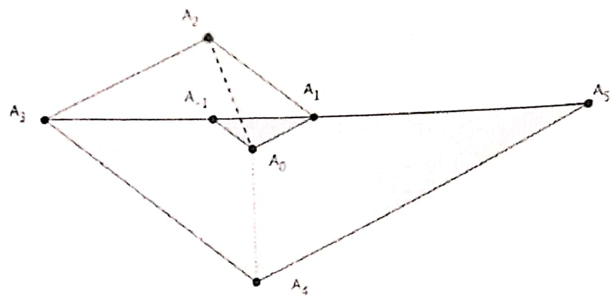


FIGURE 3.4. The construction starting from a non continue triangle

4. (r,k)- MALE SPIRAL AND LOGARITHMIC SPIRALS

Let $r \in (1,\Phi)$, and k be a positive integer, and consider the logarithmic spiral $\mathcal{S}$ whose equation is:

$$\rho = te^{(1/\pi)\lg(r^k)\theta} = tr^{k\theta/\pi}, \text{ where t depends on the starting point } (\rho(0),0). \tag{4.1}$$

The factor growth of $\mathcal{S}$, or more simply the *growth of* $\mathcal{S}$, is the term r^{2k}.

The next result shows that the (r,k)-male spiral is connected to a pair of logarithmic spirals: $\mathcal{S}_1 = \mathcal{S}_1(r,k)$ and $\mathcal{S}_2 = \mathcal{S}_2(r,k)$.

Theorem 4.1. *Let $r \in (1,\Phi)$, and k be a positive integer. Then all the vertices A_{1-2n} (indexed by odd numbers) of the (r,k)-male spiral $\mathcal{P}_{r,k}$, lie on a logarithmic spiral $\mathcal{S}_1$ of growth r^{2k}, with starting point $A_1 = (\frac{r^2}{r^k+1}, 0)$, and all the even points A_{2n} lie on a logarithmic spiral $\mathcal{S}_2$ of the same growth r^{2k}, with a suitable starting point H depending on r and k.*

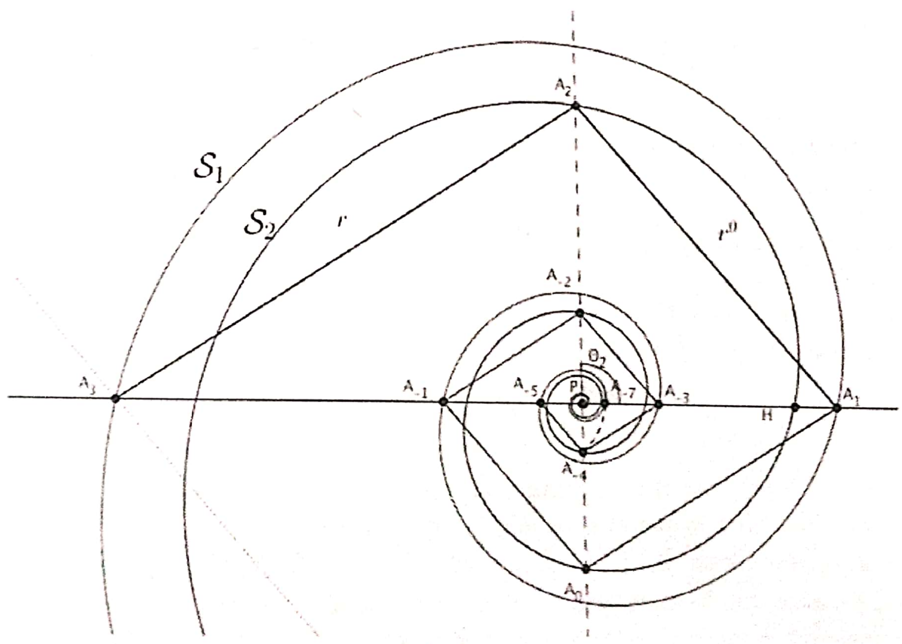


FIGURE 4.1. Logarithmic spirals associated with the (r,k)-male spiral

Proof. By virtue of Corollary 3.2, for every $m \in \mathbb{Z}$, the distance of any A_m from the pole P of $\mathcal{P}_{r,k}$ increases by a factor of r^{2k} for each (positive) turn around the pole.

From item (b) of Lemma 3.1, $|\overline{A_1 P}| = \frac{r^2}{r^k+1}$, so that the equation

(4.2) $$\rho(\theta) = \frac{r^2}{r^k+1} e^{(1/\pi)\lg(r^k)\theta} = \frac{r^2}{r^k+1} r^{k\theta/\pi},$$

defines a logarithmic spiral $\mathcal{S}_1$ to which all the vertices A_{1-2n} of the (r,k)-male spiral belong.

In order to determine the equation of the logarithmic spiral $\mathcal{S}_2$ to which all the A_{2n} of the (r,k)-male spiral belong, it suffices to determine the constant t' in the equation:

(4.3) $$\rho(\theta) = t' e^{(1/\pi)\lg(r^k)\theta} = t' r^{k\theta/\pi}.$$

From item (c) of Lemma 3.1 we have $|\overline{A_2 P}| = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$.

As an application of Carnot' Theorem, we will determine that angle $\theta_2 := A_1\hat{P}A_2$: $|A_2A_1|^2 = |PA_2|^2 + |PA_1|^2 - 2|PA_2||PA_1|cos(\theta_2)$. It follows

$$cos(\theta_2) = \frac{|PA_2|^2 + |PA_1|^2 - |A_2A_1|^2}{2|PA_2||PA_1|}$$

$$= \left(\frac{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}{(r^k+1)^2} + \frac{r^4}{(r^k+1)^2} - 1\right)\frac{1}{2}\frac{(r^k+1)}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}\frac{r^k+1}{r^2}$$

$$= \frac{1}{2r^2}\frac{-r^{k+4}-r^k+r^{k+2}+r^4+r^2-1}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}.$$

Thus

(4.4) $$\theta_2 = arccos\left(\frac{1}{2r^2}\frac{-r^{k+4}-r^k+r^{k+2}+r^4+r^2-1}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}\right).$$

Now we are in a position to determine t' in equation $\rho = t'e^{(h\theta)}$, which describes this new spiral. For this purpose, we substitute θ_2 in equation 4.3, and we have:

$$|PA_2| = \rho(\theta_2) = t' r^{k\theta_2/\pi} \text{ and hence; } t' = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\frac{1}{r^{k\theta_2/\pi}}.$$

Thus the equation of logarithmic spiral $\mathcal{S}_2$ also described by vertices A_{2n} of the (r,k)-male is:

(4.5) $$\rho(\theta) = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\frac{1}{r^{k\theta_2/\pi}} r^{k\theta/\pi}, \quad \text{where } \theta_2 \text{ is given by equation 4.4}$$

Thus the starting point of $\mathcal{S}_2$ is $H = (\frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\frac{1}{r^{k\theta_2/\pi}}, 0)$, □

The following remarks follow from Theorem 4.1:

Remark 4.1. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 2) = \mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 2)$ is the same spira Solaris, Indeed it can be checked that $A_1 = H \equiv (1/\Phi, 0)$.

Remark 4.2. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 1)$ and $\mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 1)$ are two Pheidia Spirals, with different starting points.

Remark 4.3. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 4)$ and $\mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 4)$ are two Golden Spirals, with different starting points.

Remark 4.4. Let $\mathcal{P}_{r,k}$ be an (r,k)-male spiral. If $r \neq \sqrt{\Phi}$, there is no logarithmic spiral through all the vertices of $\mathcal{P}_{r,k}$.

5. Conclusions

Note that starting from an assigned segment $\mathbf{r} = \overline{A_1A_2}$ of length $r \in (1/\Phi, \Phi)$, the construction that we have presented (see 3.1) can be made by ruler and compass. Infact all the powers r^n, and all the n-parts of $\mathbf{r}$ can be constructed, so that it is possible to construct any (r,k)-mail spiral, for every positive integer k.
In the first section we have remarked that logarithmic spirals appear in nature, and some of them can be discretized by triangles. On the other hand, we have seen that every (r,k)-male spiral is connected to a pair of logarithmic spirals: $\mathcal{S}_1 = \mathcal{S}_1(r,k)$ and $\mathcal{S}_2 = \mathcal{S}_2(r,k)$. Conversely, here we may highlight that every logarithmic spiral can be discretized by an (r,k)-male spiral. To this end, it is enough to see that

$$\forall h \in \mathbb{R}^+ \quad \lim_n \sqrt[n]{e^{h\pi}} = 1, \tag{5.1}$$

so that there exist infinite pairs $(k, \sqrt[k]{e^{h\pi}})$, such that k is a positive integer and $\sqrt[k]{e^{h\pi}}$ lies in $(1, \Phi)$. Therefore, if $\mathcal{S}$ is a logarithmic spiral, then we can write its equation as follows:

$$\rho = te^{h\theta} = tr^{\frac{k\theta}{\pi}} \quad \text{where} \quad r = \sqrt[k]{e^{h\pi}}. \tag{5.2}$$

An application of Theorem 4.1 shows that the odd vertices (resp. the even vertices) of the $(\sqrt[k]{e^{h\pi}}, k)$-male spiral discretize $\mathcal{S}$.

We note that the discretization of logarithmic spirals provided by Theorem 4.1 has a low implementation difficulty, as the vertices of the male spiral are easily computable (see Lemma 3.1 and Corollary 3.2). This could help the study of all those phenomena related to logarithmic spirals.
The property of *self-similarity* of a logarithmic spiral leads to the consideration of these curves as a model in the study of the Fractal process. This connection is related for example to the notion of *dissipative quantum interference phase* (see [20]).
The use of the discretization of the curve can be useful in the learning of the Digital design Process (see [17], Introduction and Part II) which is an important skill for the experience of an architect. The discretization of a logarithmic spiral also has applications in seismology for the prevention of collapse mechanisms (see [10]).

It is natural to ask whether there exist some kind of "good" spirals through all the vertices of an (r,k)-male spiral, $\mathcal{P}_{r,k}$. Our conjecture is that the answer is positive.
We will define the *elliptic logarithmic spiral* and we denote it by $\mathcal{E}$, the curve whose equation is:

$$\begin{cases} x(\theta) = t_1 r^{h\theta/\pi} cos(\theta) \\ y(\theta) = t_2 r^{h\theta/\pi} sin(\theta), \end{cases} \quad \text{where } t_1 \text{ and } t_2 \text{ depend on the initial condition.} \tag{5.3}$$

Note that when $t_1 = t_2$ we have the equation of a logarithmic spiral, so we may think of $\mathcal{E}$ as an elliptic curve with an exponential growth around its pole P. In general $\mathcal{E}$ is not equiangular like the logarithmic spirals; on the other hand it preserves the proportion growth.

The problem is: given $r \in (1, \Phi)$, and a positive integer k, can we determine h, t_1, t_2 depending on r and k, such that the equation 5.3 is satisfied? In other words: is there an elliptic logarithmic spiral, $\mathcal{E}_{r,k}$, through all the vertices of the (r, k)-spiral?

We represent the problem by a figure:

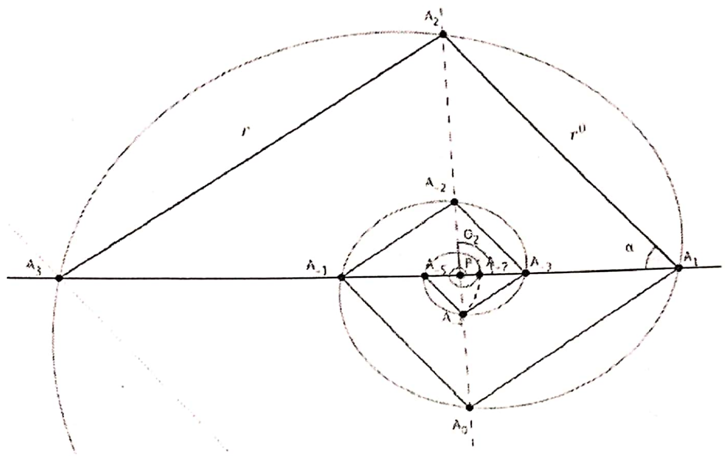


FIGURE 5.1. Elliptic logarithmic spiral approximately through all vertices of an assigned (1.35, 2)-male spiral

Note that this last family of spirals also occurs in nature. It is known (see [6] p. 113) that the two-thirds of all galaxies have a spiral structure. Among these there are some, for example the "S0 type" galaxies, that are placed in between the elliptical and logarithmic spiral galaxies. (see [16], page 390). Also in Architecture the elliptic logarithmic spirals could be employed to realize special curves and surfaces (see for example [21], [22] and [23]). Here their discretization by triangles suggest important informations.

Examples of the arc length parametrization - 2 - Catenary and Cycloid

321901076 Koichi Kato

May,4, 2020

This is related to

- Chapter 2 in Calculus II
- Chapter 2 in O'Neill

Example 3 (Catenary) Let a and c be positive numbers. Let $f : [-c, c] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = (t, a\cosh(\frac{t}{a}))$ for t in $[-c, c] \subset \mathbb{R}$. f is a function of class C^1 on $(-c, c)$. The corresponding regular curve $f((-c, c))$ is called Catenary (FIG.1). We calculate here the length L_f of the curve in [0,c].

$$L_f := \int_0^c \|f'(t)\|dt = \int_0^c \|(1, \sinh(\frac{t}{a}))\|dt = \int_0^c (1 + (\sinh(\frac{t}{a}))^2)^{1/2}dt$$

(Since $\cosh(\frac{t}{a}) > 0$ on $(0, c)$)

$$= \int_0^c \cosh(\frac{t}{a})dt = a\sinh(\frac{c}{a}).$$

Let $\psi : (0, c) \to \mathbb{R}$ be a function given for any $t \in (0, c)$ by

$$\psi(t) := \int_0^t \|f'(s)\|ds = a\sinh(\frac{t}{a}).$$

ψ is strictly increasing on $[0, c]$ and differentiable (FIG.1), with $\psi'(t) = \|f'(t)\| = \cosh(\frac{t}{a}) > 0$ for any $t \in (0, c)$. ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [0, c]$.

In this example,

$$s = \psi(t) = a\sinh(\frac{t}{a}) = a\frac{e^{\frac{t}{a}} - e^{-\frac{t}{a}}}{2},$$

$$\frac{2s}{a} = e^{\frac{t}{a}} - e^{-\frac{t}{a}},$$

$$e^{2\frac{t}{a}} - \frac{2s}{a}e^{\frac{t}{a}} - 1 = 0,$$

$$e^{\frac{t}{a}} = \frac{s}{a} \pm \left(\frac{s^2}{a^2} + 1\right)^{1/2} = \frac{s \pm \sqrt{s^2 + a^2}}{a}.$$

Since $e^{\frac{t}{a}} > 0$, the right hand side must be $\frac{s + \sqrt{s^2 + a^2}}{a}$.

Hence $\frac{t}{a} = \ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right)$. That is, $t = a\ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right)$.

Thus, if we set

$$\varphi : [0, L_f] \to [0, c], \quad \varphi(s) := \psi^{-1}(s) = a\ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$, and the composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = \left(a\ln(\frac{s + \sqrt{s^2 + a^2}}{a}), \ a\cosh(\ln(\frac{s + \sqrt{s^2 + a^2}}{a}))\right).$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|},$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\|f'(\varphi(s))\frac{1}{\|f'(\varphi(s))\|}\right\| = \frac{1}{\|f'(\varphi(s))\|}\|f'(\varphi(s))\| = 1.$$

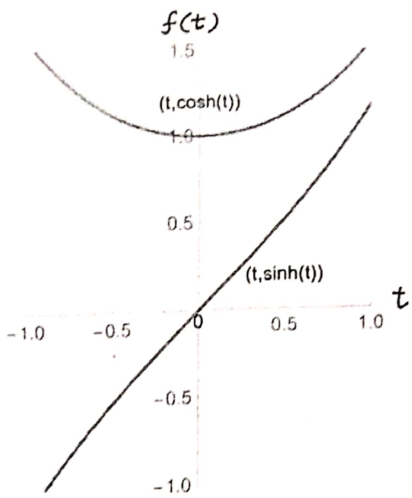


FIG. 1 Graphs of cosh(t) (Catenary) and sinh(t) for $a = 1$.

□

Example 4 (Cycloid) Let r be a positive number. Let $f : [0, 2\pi] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = (-r\sin(t) + rt, -r\cos(t) + r)$ for t in $[0, 2\pi]$. f is a function of class C^1 on $(0, 2\pi)$. The corresponding regular curve $f((0, 2\pi))$ is called Cycloid (FIG.2).

$$f'(t) = (-r\cos(t) + r, r\sin(t)),$$

$$\|f'(t)\| = \sqrt{(-r\cos(t) + r)^2 + (r\sin(t))^2} = \sqrt{2r^2 - 2r^2\cos(t)} = \sqrt{2}r\sqrt{1 - \cos(t)}$$

$$= \sqrt{2}r\sqrt{1 - (2\cos^2(\frac{t}{2}) - 1)} = \sqrt{2}r\sqrt{2 - 2\cos^2(\frac{t}{2})} = 2r\sqrt{1 - \cos^2(\frac{t}{2})}$$

(Since $\sin(\frac{t}{2}) > 0$ on $(0, 2\pi)$)

$$= 2r\sin(\frac{t}{2}).$$

The length L_f of the curve is

$$L_f := \int_0^{2\pi} \|f'(t)\| dt = \int_0^{2\pi} 2r\sin(\frac{t}{2}) dt = (-4r)\cos(\frac{t}{2})\Big|_{t=0}^{t=2\pi} = (-4r)(-1 - 1) = 8r.$$

Let $\psi : (0, 2\pi) \to \mathbb{R}$ be a function given for any $t \in (0, 2\pi)$ by

$$\psi(t) := \int_0^t \|f'(s)\| ds = (-4r)\cos(\frac{s}{2})\Big|_{s=0}^{t} = (-4r)\cos(\frac{t}{2}) + 4r.$$

ψ is strictly increasing on $[0, 2\pi]$ and differentiable, with

$$\psi'(t) = \|f'(t)\| = 2r\sin(\frac{t}{2}) > 0 \quad \forall t \in (0, 2\pi).$$

ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [0, 2\pi]$.

In this example,

$$s = \psi(t) = (-4r)\cos(\frac{t}{2}) + 4r,$$

$$\cos(\frac{t}{2}) = \frac{s-4r}{-4r} = 1 - \frac{s}{4r},$$

$$\frac{t}{2} = \arccos(1 - \frac{s}{4r}),$$

$$t = \psi^{-1}(s) = 2\arccos(1 - \frac{s}{4r})$$

Thus, if we set

$$\varphi : [0, L_f] \to [0, 2\pi], \quad \varphi(s) := \psi^{-1}(s) = 2\arccos(1 - \frac{s}{4r}),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$ (FIG.3). The composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = (-r\sin(\varphi(s)) + r\varphi(s), -r\cos(\varphi(s)) + r)$$

$$= \left(-r\sin(2\arccos(1 - \frac{s}{4r})) + r(2\arccos(1 - \frac{s}{4r})), -r\cos((2\arccos(1 - \frac{s}{4r})) + r\right).$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|},$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\| f'(\varphi(s))\frac{1}{\|f'(\varphi(s))\|} \right\| = \frac{1}{\|f'(\varphi(s))\|}\|f'(\varphi(s))\| = 1.$$

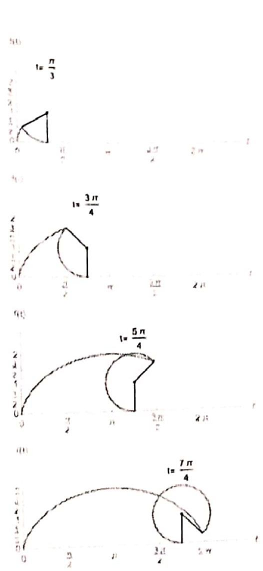


FIG. 2 drawing a Cycloid for r=1.

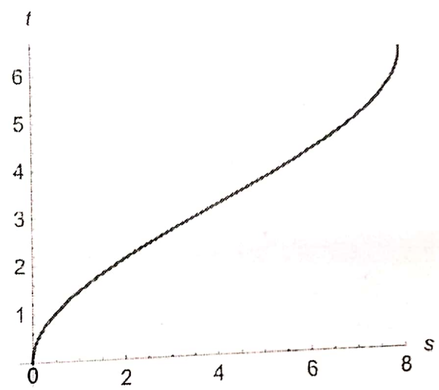


FIG. 3 t(s)=2 arccos(1-s/4), when r=1.

□

References

[1] Wikipedia, Arc length.

[2] Mathematics Stack Exchange, "Need help to parametrize the catenary by arc length".

[3] wwwf.imperial.ac.uk/metric/ ..., "Hyperbolic Functions: Inverses".

Unit - 4

Partial Derivatives

Partial Derivatives

Just as derivatives can be used to explore the properties of functions of 1 variable, so also derivatives can be used to explore functions of 2 variables. In this section, we begin that exploration by introducing the concept of a *partial derivative* of a function of 2 variables.

In particular, we define the *partial derivative* of $f(x, y)$ with respect to x to be

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

when the limit exists. That is, we compute the derivative of $f(x, y)$ as if x is the variable and all other variables are held constant. To facilitate the computation of partial derivatives, we define the operator

$$\frac{\partial}{\partial x} = \text{"The partial derivative with respect to } x\text{"}$$

Alternatively, other notations for the partial derivative of f with respect to x are

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \partial_x f(x, y)$$

EXAMPLE 1 Evaluate f_x when $f(x, y) = x^2 y + y^2$.

Solution: To do so, we write

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 y + y^2\right) = \frac{\partial}{\partial x} x^2 y + \frac{\partial}{\partial x} y^2$$

and then we evaluate the derivative as if y is a constant. In particular,

$$f_x(x, y) = y \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial x} y^2 = y \cdot 2x + 0$$

That is, y factors to the front since it is considered constant with respect to x. Likewise, y^2 is considered constant with respect to x, so that its derivative with respect to x is 0. Thus, $f_x(x, y) = 2xy$.

Likewise, the *partial derivative* of $f(x, y)$ *with respect to* y is defined

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

when the limit exists. That is, we evaluate f_y as if y is varying and all other quantities are constant. Moreover, we also define the operator

$$\frac{\partial}{\partial y} = \text{"The partial derivative with respect to } y\text{"}$$

and we often use other notations for $f_y(x, y)$:

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \partial_y f(x, y)$$

EXAMPLE 2 Find f_x and f_y when $f(x, y) = y \sin(xy)$

Solution: To find f_x, we use the chain rule

$$f_x = y \frac{\partial}{\partial x} \sin(xy) = y \cos(xy) \frac{\partial}{\partial x} xy = y^2 \cos(xy)$$

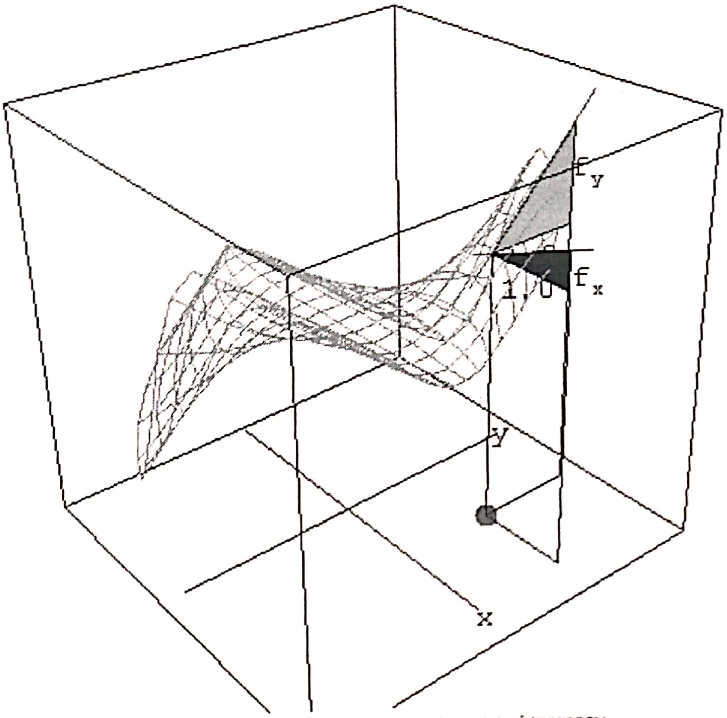
However, to find f_y, we begin with the product rule:

$$f_y = \frac{\partial}{\partial y} [y \sin(xy)] = \left(\frac{\partial}{\partial y} y\right) \sin(xy) + y \frac{\partial}{\partial y} \sin(xy)$$

We then use the chain rule to evaluate $\partial_y \sin(xy)$:

$$\begin{aligned} f_y &= \sin(xy) + y \cos(xy) \frac{\partial}{\partial y}(xy) \\ &= \sin(xy) + xy \cos(xy) \end{aligned}$$

If $y = q$ for some constant q, then $f(x, q)$ is a function of x, and $f_x(x, q)$ is the slope of the tangent line to the curve $z = f(x, q)$ in the $y = q$ plane. Similarly, $f_y(p, y)$ for p constant is the slope of a tangent line to the curve $z = f(p, y)$ in the $x = p$ plane.



Drag to rotate. Drag red point to change the point of tangency
Shift-drag to resize or to tilt z-axis.

That is, $f_x(x, y)$ is the slope of a tangent line to $z = f(x, y)$ parallel to the xz-plane, while $f_y(x, y)$ is the slope of a tangent line to $z = f(x, y)$ in the yz-plane, an idea we will explore more fully in a later section.

blue*EXAMPLE 3* blackFind f_x and f_y when

$$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

Solution: To find f_x, we begin with the chain rule

$$f_x = \frac{\partial}{\partial x}\tan^{-1}(\text{input}) = \frac{1}{(\text{input})^2 + 1}\frac{\partial}{\partial x}(\text{input})$$

where the input is y/x. Writing the input as $x^{-1}y$ and substituting then yields

$$f_x = \frac{1}{(x^{-1}y)^2 + 1}\frac{\partial}{\partial x}\left(x^{-1}y\right) = \frac{-x^{-2}y}{x^{-2}y^{-2} + 1}$$

To simplify this expression, we multiply the numerator and denominator by x^2:

$$f_x = \frac{x^2\left(-x^{-2}y\right)}{x^2\left(x^{-2}y^{-2}+1\right)} = \frac{-x^2x^{-2}y}{x^2x^{-2}y^2+x^2} = \frac{-y}{y^2+x^2}$$

To find f_y, we again begin with the chain rule,

$$f_y = \frac{\partial}{\partial y}\tan^{-1}(\text{input}) = \frac{1}{(\text{input})^2+1}\frac{\partial}{\partial y}(\text{input})$$

where the input is y/x. The result is that

$$f_y = \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\left[\frac{1}{x}\frac{\partial}{\partial y}(y)\right]$$

which simplifies to

$$f_y = \frac{1}{\left(\frac{y^2}{x^2}+1\right)x} = \frac{x}{\left(\frac{y^2}{x^2}+1\right)x^2} = \frac{x}{x^2+y^2}$$

Check your Reading: What happend to the expression x^2x^{-2} in example 3?

Interpretations of the Partial Derivative

Analogous to a function of 2 variables, we define a *function of 3 variables* is a mapping that assigns one and only one real number to each point in a subset of 3 dimensional space. It follows that a function of three variables is of the form

$$F(x,y,z) = \text{“expression in } x, y, \text{ and } z\text{”}$$

Partial derivatives of functions of 3 variables are defined analogously to partial derivatives of functions of two variables (see the exercises). Thus, to differentiate a function of 3 variables $F(x,y,z)$ with respect to x, we differentiate as if y and z are constants. The partial derivatives F_y and F_z are defined similarly , and correspondingly, to calculate F_y and F_z, we differentiate with respect to y and z, respectively.

EXAMPLE 4 Find the first partial derivatives of $F(x,y,z) = x^3 + 3xyz + z^2$.

Solution: To compute f_x, we treat y and z as if they were constant:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}\left(x^3+3xyz+z^2\right) \\ &= \frac{\partial}{\partial x}x^3+3yz\frac{\partial}{\partial x}x+\frac{\partial}{\partial x}z^2 \\ &= 3x^2+3yz \end{aligned}$$

Likewise, F_y follows from treating x and z like constants,

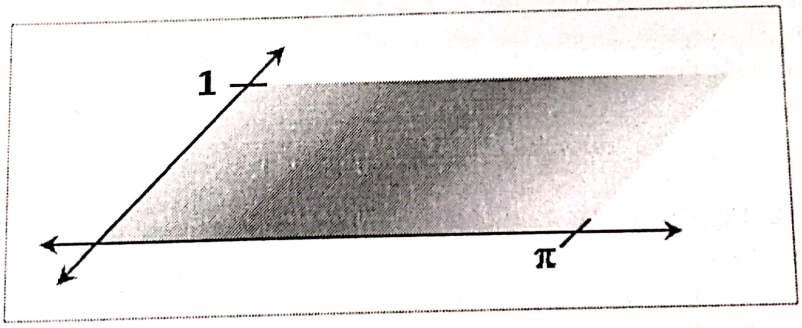
$$F_y=\frac{\partial}{\partial y}\left(x^3+3xyz+z^2\right)=0+3xz+0=3xz$$

and F_z follows from treating x and y like constants.

$$F_z=\frac{\partial}{\partial z}\left(x^3+3xyz+z^2\right)=3xy+2z$$

In many applications, functions of three variables occur in the form $u(x,y,t)$, where t is a measure of *time*. In such examples, the definition of the partial derivative u_t implies that it is *the rate of change* of $u(x,y,t)$ with respect to t. Indeed, partial derivatives often occur in applications as a rate of change of a given output with respect to only one of several inputs.

EXAMPLE 5 A rectangular sheet of metal with a length of π feet and a width of 1 foot has its left section placed in an oven and its rightmost extent placed in liquid nitrogen.



Upon being removed from the oven and nitrogen, its temperature u in $°F$ at time t in seconds and at a point (x,y) on the sheet is given by

$$u(x,y,t)=75+300e^{-0.2t}\cos(x)\cosh(y)$$

How fast is the temperature of the sheet changing with respect to time at the point $(0,0)$? At the point $(\pi,0)$? How do the rates compare?

Solution: The rate of change of u with respect to t is given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(75 + 300e^{-0.2t}\cos(x)\cosh(y)\right) = -60e^{-0.2t}\cos(x)\cosh(y)$$

Thus, at $(0,0)$, the time rate of change of temperature is

$$\frac{\partial u}{\partial t}(0,0,t) = -60e^{-0.2t}\cos(0)\cosh(0) = -60e^{-0.2t}\ {}^\circ F\ per\ sec$$

while at $(\pi,0)$, the time rate of change of temperature is

$$\frac{\partial u}{\partial t}(\pi,0,t) = -60e^{-0.2t}\cos(\pi)\cosh(0) = 60e^{-0.2t}\ {}^\circ F\ per\ sec$$

This means that the temperature increases to $75^\circ F$ over time at the point $(\pi,0)$, while it decreases down to $75^\circ F$ at $(0,0)$. **Note:** this fits well with the fact that the temperature initially at $(0,0)$ is $375^\circ F$, while the tmperature initially at $(\pi,0)$ is $-225^\circ F$.

Check your Reading: After a long period of time, what will be the approximate temperature of the metal sheet at every point on the sheet?

Second Derivatives

The second partial derivative of f with respect to x is denoted f_{xx} and is defined

$$f_{xx}(x,y) = \frac{\partial}{\partial x} f_x(x,y)$$

That is, f_{xx} is the derivative of the first partial derivative f_x. Likewise, the second partial derivative of f with respect to y is denoted f_{yy} and is defined

$$f_{yy}(x,y) = \frac{\partial}{\partial y} f_y(x,y)$$

Finally, the *mixed* partial derivatives are denoted f_{xy} and f_{yx}, respectively, and are defined

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y) \qquad \text{and} \qquad f_{yx} = \frac{\partial}{\partial x} f_y(x,y)$$

Collectively, f_{xx}, f_{yy}, f_{xy}, and f_{yx} are known as the *second partial derivatives* of $f(x,y)$. Moreover, we sometimes denote the second partial derivatives in the form

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

EXAMPLE 6 Find the second partial derivatives of

$$f(x,y) = x^3 + 3x^2y^2$$

Solution: The first partial derivatives are $f_x = 3x^2 + 6xy^2$ and $f_y = 6x^2y$. As a result, we have

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x} f_x(x,y) = \frac{\partial}{\partial x}\left(3x^2+6xy^2\right) = 6x + 6y^2 \\ f_{yy}(x,y) &= \frac{\partial}{\partial y} f_y(x,y) = \frac{\partial}{\partial y}\left(6x^2y\right) = 6x^2 \\ f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial}{\partial y}\left(3x^2+6xy^2\right) = 12xy \\ f_{yx}(x,y) &= \frac{\partial}{\partial x} f_y(x,y) = \frac{\partial}{\partial x}\left(6x^2y\right) = 12xy \end{aligned}$$

Notice that the mixed partial derivatives are the same. Indeed, the mixed partials are always the same for "nice" functions, as is stated below in *Clairaut's theorem.*

Clairaut's Theorem: If f is defined on a neighborhood of (p,q) and if f_{xy} and f_{yx} are continuous on that neighborhood, then

$$f_{xy}(p,q) = f_{yx}(p,q)$$

There are functions that do not satisfy the hypotheses of theorem 3.1 for which the mixed partials are not the same at some point (see exercise 46). However, our focus will be on functions with continuous second partial derivatives, in which case the mixed partials are the same (a proof of Clairaut's theorem is given in chapter 4).

EXAMPLE 7 Find f_{yx} and f_{xy} for $f(x,y) = x\sin(xy)$

Solution: The first partial derivatives are

$$f_x = \sin(xy) + xy\cos(xy) \qquad \text{and} \qquad f_y = x^2\cos(xy)$$

The product rule thus implies that

$$f_{yx} = \frac{\partial}{\partial x}\left(x^2\cos(xy)\right) = 2x\cos(xy) - x^2y\sin(xy)$$

Now let's compute f_{xy} (and thus confirm theorem 3.1):

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(\sin(xy) + xy\cos(xy)\right) \\ &= x\cos(xy) + x\cos(xy) + xy\left(-\sin(xy)\frac{\partial}{\partial y}xy\right) \\ &= 2x\cos(xy) - x^2 y\sin(xy) \end{aligned}$$

Check your Reading: If $f(x, y, z)$ is infinitely differentiable in each variable, then is $f_{xz} = f_{zx}$?

Higher Derivatives

Higher partial derivatives are defined similarly. For example, the third derivative of f with respect to x is the partial derivative with respect to x of the second derivative f_{xx}. That is,

$$f_{xxx}(x, y) = \frac{\partial}{\partial x} f_{xx}(x, y)$$

Similarly, f_{xxy} is defined

$$f_{xxy}(x, y) = \frac{\partial}{\partial y} f_{xx}(x, y)$$

In operator notation, the partial derivative of f for m times with respect to x and n times with respect to y is denoted by

$$\frac{\partial^{m+n} f}{\partial x^m \, \partial y^n}$$

The $m+n$ partial derivatives of $f(x, y)$ are then defined in terms of the previous partial derivatives as

$$\frac{\partial^{m+n} f}{\partial x^m \, \partial y^n} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^{m+n-2} f}{\partial x^{m-1} \partial y^{n-1}}\right)$$

when $f(x, y)$ and its partial derivatives are continuous on a region through the $m + n$ order.

EXAMPLE 8 Find f_{xxyy} if $f(x,y) = x^4y^4$.

Solution: It is easy to show that $f_{xx} = 12x^2y^4$. Thus,

$$f_{xxy} = \frac{\partial}{\partial y} f_{xx} = \frac{\partial}{\partial y} 12x^2y^4 = 48x^2y^3$$

and similarly,

$$f_{xxyy} = \frac{\partial}{\partial y} f_{xxy} = \frac{\partial}{\partial y} 48x^2y^3 = 144x^2y^2$$

Moreover, we usually assume that f is sufficiently smooth at all points where partial derivatives are defined so that mixed partials are independent of the order of differentiation. Indeed, notice that if $f(x,y) = x^4y^4$, then

$$f_{xyx} = \frac{\partial}{\partial x} f_{xy} = \frac{\partial}{\partial x} 16x^3y^3 = 48x^2y^3$$

which is the same as f_{xxy} in example 8. In addition, $f_{xyxy} = 144x^2y^2 = f_{xxyy}$.

Exercises

Find $f_x(x,y)$ and $f_y(x,y)$ for each of the following:

1. $f(x,y) = x^2 + y^3$
2. $f(x,y) = x^2 + 2xy + y^3$
3. $f(x,y) = (x+2y)^2$
4. $f(x,y) = (x^2+2y)^2$
5. $f(x,y) = x\sin(y)$
6. $f(x,y) = e^x \ln(y^2+1)$
7. $f(x,y) = \exp(-x^2 - y^2)$
8. $f(x,y) = \tan^{-1}(xy)$
9. $f(x,y) = x\cos(xy)$
10. $f(x,y) = x\sin(xy)$
11. $f(x,y) = y^x$
12. $f(x,y) = x^y$
13. $f(x,y) = \frac{x}{x^2+y^2}$
14. $f(x,y) = \int_x^y \sin(t^2)\,dt$

Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} for each of the following. Then show that the mixed partials are the same.

15. $f(x,y) = x^2 + y^3$
16. $f(x,y) = x^2 + 2xy + y^3$
17. $f(x,y) = (x+2y)^2$
18. $f(x,y) = (x^2+2y)^2$
19. $f(x,y) = x\sin(y)$
20. $f(x,y) = e^x \ln(y^2+1)$
21. $f(x,y) = x\cos(xy)$
22. $f(x,y) = \tan^{-1}(xy)$
23. $f(x,y) = y^x$
24. $f(x,y) = x^y$

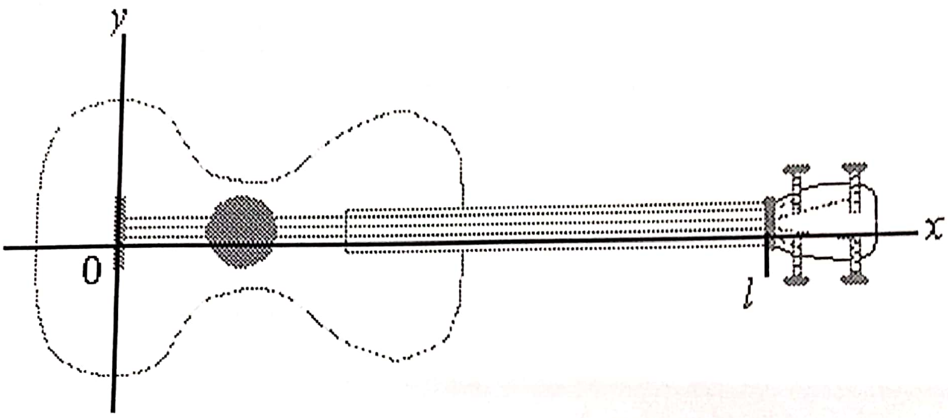
Find the indicated derivative of the given function:

25. f_{xxy} *for* $f(x,y) = x^2 + y^3$
26. f_{xxy} *for* $f(x,y) = x^2 + 2xy + y^3$
27. f_{xyx} *for* $f(x,y) = (x+2y)^2$
28. f_{yxy} *for* $f(x,y) = (x^2+2y)^2$
29. f_{xxyy} *for* $f(x,y) = x\sin(y)$
30. $f_{xxxxxxxy}$ *for* $f(x,y) = e^x \ln(y^2+1)$
31. f_{xxxy} *for* $f(x,y) = y\cos(xy)$
32. f_{xyy} *for* $f(x,y) = \sin(x)\tan(xy)$

33. A vibrating string has a displacement $y = u(x,t)$ in cm at a distance x in cm from one end and at time t in seconds, where

$$u(x,t) = 2\sin(120\pi(x-t))$$

How fast (in units of cm per sec) is the string vibrating at a horizontal distance $x = 1.2$ cm from one end at time $t = 2$ seconds? At time $t = 3$ seconds?
34. Suppose that a string is attached at its endpoints $x = 0$ and $x = l$, for some number l.



Suppose also that $y = u(x,t)$ models the displacement at x in $[0,l]$ of the string at time t, where

$$u(x,t) = A\cos(at)\sin\left(\frac{\pi}{l}x\right)$$

with A and a constant.

1. (a) Show that $u(0,t) = u(l,t) = 0$ for all t. How does this relate to $u(x,t)$ being a model of a string?
   (b) What is the rate of change of $u(x,t)$ at $x = l/2$ at any given time?

35. The function $u(x,t) = e^{-t}\sin^2(\pi x) + 32$ models the temperature in $°F$ of a 1 foot long thin rod in which both ends are held at the freezing point at all times t. How fast is the temperature decreasing at the midpoint of the rod when $t = 0$? When $t = 1$? When $t = 2$?
36. The function $u(x,y,t) = 2\sin(3x)\sin(4y)\cos(5t)$ models the displacement u in cm of a vibrating rectangular membrane at time t in seconds and at

a point (x, y) on the membrane. How fast is the displacement of the membrane above the point $(1, 1)$ changing with respect to time at $t = 1$ seconds?

37. It can be shown that an ideal gas with fixed mass has an absolute temperature R, a pressure P, and a volume V that satisfies

$$T = kPV$$

where k is a constant. How fast does the temperature T change with respect to the volume V?

38. The total resistance R produced by two resistors with resistances R_1 and R_2, respectively, satisfies

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

What is the rate of change of the total resistance R with respect to the resistance R_1?

39. If two planets with masses M and m are located at the points (x, y, z) and $(0, 0, 0)$, respectively, then the potential energy of their mutual gravitational attraction is given by

$$\phi(x, y, z) = G\frac{Mm}{\sqrt{x^2 + y^2 + z^2}}$$

where G is the universal gravitational constant. At what rate is the potential energy changing with respect to x? With respect to y?

40. A *Cobb-Douglas* production function is a function of the form $P = bL^{\alpha}K^{\beta}$ where b, α, and β are constants. What is the rate of change of P with respect to L? With respect to P?

41. Suppose we consider $f(x, y) = x^2 + y^2$.

1. (a) What is the slope of $z = f(x, y)$ for $(p, q) = (1, 2)$ in the x-direction? in the y-direction?

   (b) What curve is formed by the intersection of the plane $y = 2$ with the surface $z = x^2 + y^2$? How does it relate to $f_x(1, 2)$?

   (c) What curve is formed by the intersection of the plane $x = 1$ with the surface $z = x^2 + y^2$? How does it relate to $f_y(1, 2)$?

42. Suppose we consider $f(x, y) = x^2 + xy$.

1. (a) What is the slope of $z = f(x, y)$ for $(p, q) = (1, 2)$ in the x-direction? in the y-direction?

   (b) What curve is formed by the intersection of the plane $y = 2$ with the surface $z = x^2 + y^2$? How does it relate to $f_x(1, 2)$?

   (c) What curve is formed by the intersection of the plane $x = 1$ with the surface $z = x^2 + y^2$? How does it relate to $f_y(1, 2)$?

43. If $f(x,y) = g(x) + h(y)$, then what is f_x and f_y? What is the equation in x and z of the curve formed by the intersection of $z = f(x,y)$ with the vertical plane $y = q$? with the vertical plane $x = p$? How are these curves related to $f_x(p,q)$ and $f_y(p,q)$, respectively?

44. If $f(x,y) = g(x) + h(y)$, then what is f_x and f_y? What is the equation in x and z of the curve formed by the intersection of $z = f(x,y)$ with the vertical plane $y = q$? with the vertical plane $x = p$? How are these curves related to $f_x(p,q)$ and $f_y(p,q)$, respectively?

45. If f_x and f_y both exist, how can the limit

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y+h)}{h}$$

be expressed in terms of the 1st partial derivatives of f?

46. Write to Learn: Write a short essay in which you use the following steps to show that

$$f(x,y) = \begin{cases} \frac{x^2y - y^2x}{x+y} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$, that $f_x(x,y)$ and $f_y(x,y)$ are continuous at $(0,0)$, but that

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

1. (a) Show that if $(x,y) \neq (0,0)$, then

$$f(x,y) = \frac{x-y}{\frac{1}{x} + \frac{1}{y}}$$

and correspondingly

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{x-y}{\frac{1}{x} + \frac{1}{y}} = 0$$

(b) Show that if $(x,y) \neq (0,0)$, then

$$f_x(x,y) = \frac{y\left(x^2 + 2xy - y^2\right)}{(x+y)^2}$$

and explain why that as a result, we have

$$\lim_{(x,y) \to (0,0)} f_x(x,y) = 0$$

(c) Define $f_x(0,0) = 0$ and then evaluate

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h}$$

(d) Repeat (b) and (c) beginning with the fact that

$$f_y(x,y) = \frac{x\left(x^2 - 2xy - y^2\right)}{(x+y)^2}$$

The outcome should be that $f_{yx}(0,0)$ is not the same as $f_{xy}(0,0)$.

Euler's Theorem

(For B.Sc./B.A. Part-I, Hons. And Subsidiary Courses of Mathematics)

Poonam Kumari
Department of Mathematics, Magadh Mahila College
Patna University

Contents

1. Homogeneous Function

A function f of two independent variables x, y is said to be a homogeneous function of degree n if it can be put in either of the following two forms :

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right), \text{ where } \phi \text{ denotes a function of } \frac{y}{x}$$

or $\quad f(tx, ty) = t^n f(x, y)$, where t is any positive real number.

Similarly, a function f of three independent variables x, y, z is said to be a homogeneous function of degree n if it can be put in either of the following two forms :

$$f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right), \text{ where } \phi \text{ denotes a function of } \frac{y}{x} \text{ and } \frac{z}{x}$$

or $\quad f(tx, ty, tz) = t^n f(x, y, z)$, where t is any positive real number.

The above definition can be extended to a function of any number of variables.

Example : The function

$$f(x, y) = \frac{x^4 + y^4}{x - y}$$

is a homogeneous function of degree 3, since

$$f(x, y) = \frac{x^4 + y^4}{x - y} = \frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1-\left(\frac{y}{x}\right)\right]} = x^3 \frac{\left[1+\left(\frac{y}{x}\right)^4\right]}{\left[1-\left(\frac{y}{x}\right)\right]} = x^3\phi\left(\frac{y}{x}\right)$$

or alternatively,

$$f(tx, ty) = \frac{(tx)^4 + (ty^4)}{tx - ty} = \frac{t^4\left(x^4 + y^4\right)}{t\left(x - y\right)} = t^3\frac{\left(x^4 + y^4\right)}{\left(x - y\right)} = t^3 f(x, y).$$

Similarly, the function

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{x + y + z}$$

is a homogeneous function of degree 2, since

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{x + y + z} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{x\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]} = x^2\frac{\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]} = x^2\phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

or alternatively,

$$f(tx,ty,tz)=\frac{(tx)^3+(ty^3)+(tz)^3}{tx+ty+tz}=\frac{t^3(x^3+y^3+z^3)}{t(x+y+z)}=t^2\frac{(x^3+y^3+z^3)}{(x+y+z)}=t^2 f(x,y,z).$$

Note: A polynomial function is a homogeneous function of degree n if all of its terms are of the same degree n.

Proof : Let f be a polynomial function in two independent variables x, y, i.e.,

$$f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots\dots\dots+a_{n-1}xy^{n-1}+a_ny^n.$$

Then $$f(x,y)=x^n\left[a_0+a_1\frac{y}{x}+a_2\frac{y^2}{x^2}+\dots\dots\dots+a_{n-1}\frac{y^{n-1}}{x^{n-1}}+a_n\frac{y^n}{x^n}\right]$$

$$=x^n\left[a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+\dots\dots\dots+a_{n-1}\left(\frac{y}{x}\right)^{n-1}+a_n\left(\frac{y}{x}\right)^n\right]$$

$=x^n\phi\left(\frac{y}{x}\right)$, where ϕ is a function of $\frac{y}{x}$.

Example : The function

$$f(x,y)=x^5+6x^4y+7x^3y^2+2y^5$$

is a homogeneous function of degree 5, since

$$f(x,y)=x^5+6x^4y+7x^3y^2+2y^5$$

$$=x^5\left[1+6\left(\frac{y}{x}\right)+7\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^5\right]$$

$=x^5\phi\left(\frac{y}{x}\right)$, where ϕ is a function of $\frac{y}{x}$.

Similarly, the function

$$f(x,y,z)=x^4+3x^2y^2+4xyz^2+5yz^3+6y^4+7z^4$$

is a homogeneous function of degree 4, since

$$f(x,y,z)=x^4+3x^2y^2+4xyz^2+5yz^3+6y^4+7z^4$$

$$=x^4\left[1+3\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)^2+5\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)^3+6\left(\frac{y}{x}\right)^4+7\left(\frac{z}{x}\right)^4\right]$$

$=x^4\phi\left(\frac{y}{x},\frac{z}{x}\right)$, where ϕ is a function of $\frac{y}{x}$ and $\frac{z}{x}$.

2. Euler's Theorem on Homogeneous Function of Two Variables

Statement : If u be a homogeneous function of degree n in two independent variables x, y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Proof : Let

$$u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(1)$$

where $\alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=A_1\left(\alpha_1x^{\alpha_1-1}\right)y^{\beta_1}+A_2\left(\alpha_2x^{\alpha_2-1}\right)y^{\beta_2}+A_3\left(\alpha_3x^{\alpha_3-1}\right)y^{\beta_3}+\ldots\ldots+A_n\left(\alpha_nx^{\alpha_n-1}\right)y^{\beta_n}$$

This $\Rightarrow$ $$x\frac{\partial u}{\partial x}=A_1\alpha_1x^{\alpha_1}y^{\beta_1}+A_2\alpha_2x^{\alpha_2}y^{\beta_2}+A_3\alpha_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots+A_n\alpha_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(2)$$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=A_1x^{\alpha_1}\left(\beta_1y^{\beta_1-1}\right)+A_2x^{\alpha_2}\left(\beta_2y^{\beta_2-1}\right)+A_3x^{\alpha_3}\left(\beta_3y^{\beta_3-1}\right)+\ldots\ldots+A_nx^{\alpha_n}\left(\beta_ny^{\beta_n-1}\right)$$

This $\Rightarrow$ $$y\frac{\partial u}{\partial y}=A_1\beta_1x^{\alpha_1}y^{\beta_1}+A_2\beta_2x^{\alpha_2}y^{\beta_2}+A_3\beta_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots+A_n\beta_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(3)$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(\alpha_1+\beta_1)A_1x^{\alpha_1}y^{\beta_1}+(\alpha_2+\beta_2)A_2x^{\alpha_2}y^{\beta_2}+(\alpha_3+\beta_3)A_3x^{\alpha_3}y^{\beta_3}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots+(\alpha_n+\beta_n)A_nx^{\alpha_n}y^{\beta_n}$$

$$=nA_1x^{\alpha_1}y^{\beta_1}+nA_2x^{\alpha_2}y^{\beta_2}+nA_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots\ldots\ldots+nA_nx^{\alpha_n}y^{\beta_n}$$

$$(\because\ \alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n)$$

$$=n\left(A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n}\right)$$

$=nu$ (using equation (1))

i.e., $$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Corollary : If u be a homogeneous function of degree n in two independent variables $x, y,$ then

(i) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=(n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}=(n-1)\dfrac{\partial u}{\partial y}$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u.$

Proof : (i) Since u is a homogeneous function of degree n in two independent variables $x, y,$ therefore, by Euler's Theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad \text{......(1)}$$

Differentiating both sides of equation (1) partially w. r. t. $x,$ we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}(1)+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

This $\Rightarrow x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=(n-1)\dfrac{\partial u}{\partial x}$ ....(2)

Hence (i) is proved.

(ii) Differentiating both sides of equation (1) partially w. r. t. $y,$ we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}(1)=n\frac{\partial u}{\partial y}$$

This $\Rightarrow x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial y}=n\dfrac{\partial u}{\partial y}$ $\left(\because \dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}\right)$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \qquad \text{........(3)}$$

Hence (ii) is proved.

(iii) Multiplying equations (2) and (3) by x and y respectively and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=(n-1)(nu) \quad \text{(using equation (1))}$$

i.e., $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u.$

This proves (iii).

3. Euler's Theorem on Homogeneous Function of Three Variables

Statement : If u be a homogeneous function of degree n in three independent variables $x, y, z,$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

Proof : Let

$$u = A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \qquad \ldots\ldots(1)$$

where $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_3 + \beta_3 + \gamma_3 = \ldots\ldots\ldots = \alpha_n + \beta_n + \gamma_n = n$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = A_1\left(\alpha_1 x^{\alpha_1 - 1}\right) y^{\beta_1} z^{\gamma_1} + A_2\left(\alpha_2 x^{\alpha_2 - 1}\right) y^{\beta_2} z^{\gamma_2} + A_3\left(\alpha_3 x^{\alpha_3 - 1}\right) y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n\left(\alpha_n x^{\alpha_n - 1}\right) y^{\beta_n} z^{\gamma_n}$$

This $\Rightarrow$

$$x\frac{\partial u}{\partial x} = A_1 \alpha_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 \alpha_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 \alpha_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n \alpha_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \qquad \ldots\ldots(2)$$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = A_1 x^{\alpha_1}\left(\beta_1 y^{\beta_1 - 1}\right) z^{\gamma_1} + A_2 x^{\alpha_2}\left(\beta_2 y^{\beta_2 - 1}\right) z^{\gamma_2} + A_3 x^{\alpha_3}\left(\beta_3 y^{\beta_3 - 1}\right) z^{\gamma_3} + \ldots\ldots\ldots + A_n x^{\alpha_n}\left(\beta_n y^{\beta_n - 1}\right) z^{\gamma_n}$$

This $\Rightarrow$

$$y\frac{\partial u}{\partial y} = A_1 \beta_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 \beta_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 \beta_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n \beta_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \qquad \ldots\ldots(3)$$

Similarly, differentiating both sides of equation (1) partially w. r. t. z, we get

$$\frac{\partial u}{\partial z} = A_1 x^{\alpha_1} y^{\beta_1}\left(\gamma_1 z^{\gamma_1 - 1}\right) + A_2 x^{\alpha_2} y^{\beta_2}\left(\gamma_2 z^{\gamma_2 - 1}\right) + A_3 x^{\alpha_3} y^{\beta_3}\left(\gamma_3 z^{\gamma_3 - 1}\right) + \ldots\ldots\ldots + A_n x^{\alpha_n} y^{\beta_n}\left(\gamma_n z^{\gamma_n - 1}\right)$$

This $\Rightarrow$

$$z\frac{\partial u}{\partial z} = A_1 \gamma_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 \gamma_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 \gamma_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n \gamma_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \qquad \ldots\ldots(4)$$

Adding equations (2), (3) and (4), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = (\alpha_1 + \beta_1 + \gamma_1) A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + (\alpha_2 + \beta_2 + \gamma_2) A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + (\alpha_3 + \beta_3 + \gamma_3) A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots + (\alpha_n + \beta_n + \gamma_n) A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n}$$

$$= n A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + n A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + n A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + n A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n}$$

$$(\because \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \ldots\ldots = \alpha_n + \beta_n + \gamma_n = n)$$

$$= n\left(A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots + A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n}\right)$$

$= nu$ (using equation (1))

i.e.,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$$

Example 1 : Verify Euler's Theorem when $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$.

Solution : According to Euler's Theorem, if u be a homogeneous function of degree n in two independent variables x, y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Given that

$$u = \frac{x(x^3 - y^3)}{x^3 + y^3} \qquad .....(1)$$

i.e., $$u = \frac{x^4\left[1 - \left(\frac{y}{x}\right)^3\right]}{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]} = x\frac{\left[1 - \left(\frac{y}{x}\right)^3\right]}{\left[1 + \left(\frac{y}{x}\right)^3\right]} = x\,\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

This $\Rightarrow$ The given function u is a homogeneous function of degree 1 in two independent variables x, y. Therefore Euler's Theorem will be verified if we can prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

Taking logarithm of both sides of equation (1), we get

$$\log u = \log x + \log(x^3 - y^3) - \log(x^3 + y^3) \qquad ....(2)$$

Now, differentiating both sides of equation (2) partially w. r. t. x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{1}{x^3 - y^3}(3x^2) - \frac{1}{x^3 + y^3}(3x^2)$$

This $\Rightarrow$ $$\frac{1}{u}\left(x\frac{\partial u}{\partial x}\right) = 1 + \frac{3x^3}{x^3 - y^3} - \frac{3x^3}{x^3 + y^3} \qquad ....(3)$$

Similarly, differentiating both sides of equation (2) partially w. r. t. y, we get

$$\frac{1}{u}\frac{\partial u}{\partial y} = 0 + \frac{1}{x^3 - y^3}(-3y^2) - \frac{1}{x^3 + y^3}(3y^2)$$

This $\Rightarrow$ $$\frac{1}{u}\left(y\frac{\partial u}{\partial y}\right) = -\frac{3y^3}{x^3 - y^3} - \frac{3y^3}{x^3 + y^3} \qquad ....(4)$$

Adding equations (3) and (4), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 1 + \frac{3(x^3 - y^3)}{x^3 - y^3} - \frac{3(x^3 + y^3)}{x^3 + y^3}$$

$$= 1 + 3 - 3$$

$$= 1$$

This $\Rightarrow$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

$\Rightarrow$ Euler's Theorem is verified for the given function.

Example 2 : Verify Euler's Theorem when $u = \dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$.

Solution : According to Euler's Theorem, if u be a homogeneous function of degree n in two independent variables x, y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Given that

$$u = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}} \qquad .....(1)$$

i.e.,
$$u = \frac{x^{\frac{1}{4}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{x^{\frac{1}{5}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right]} = x^{\frac{1}{20}}\frac{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right]} = x^{\frac{1}{20}}\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

This $\Rightarrow$ The given function u is a homogeneous function of degree $\frac{1}{20}$ in two independent variables x, y. Therefore Euler's Theorem will be verified if we can prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}u.$$

Taking logarithm of both sides of equation (1), we get

$$\log u = \log\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) - \log\left(x^{\frac{1}{5}} + y^{\frac{1}{5}}\right) \qquad ....(2)$$

Now, differentiating both sides of equation (2) partially w. r. t. x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}\left(\frac{1}{4}x^{-\frac{3}{4}}\right) - \frac{1}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}\left(\frac{1}{5}x^{-\frac{4}{5}}\right)$$

This $\Rightarrow$ $$\frac{1}{u}\left(x\frac{\partial u}{\partial x}\right) = \frac{x^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)} - \frac{x^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}} + y^{\frac{1}{5}}\right)} \qquad ....(3)$$

Similarly, differentiating both sides of equation (2) partially w. r. t. y, we get

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{1}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}\left(\frac{1}{4}y^{-\frac{3}{4}}\right) - \frac{1}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}\left(\frac{1}{5}y^{-\frac{4}{5}}\right)$$

This $\Rightarrow$ $$\frac{1}{u}\left(y\frac{\partial u}{\partial y}\right) = \frac{y^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)} - \frac{y^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}} + y^{\frac{1}{5}}\right)} \qquad ....(4)$$

Adding equations (3) and (4), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}-\frac{x^{\frac{1}{5}}+y^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}$$

$$=\frac{1}{4}-\frac{1}{5}$$

$$=\frac{1}{20}$$

This $\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}u$

$\Rightarrow$ Euler's Theorem is verified for the given function.

Example 3 : If $u=\tan^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u.$

Solution : Given that

$$u=\tan^{-1}\frac{x^2+y^2}{x+y}.$$

This $\Rightarrow \tan u=\frac{x^2+y^2}{x+y}=\frac{x^2\left[1++\left(\frac{y}{x}\right)^2\right]}{x\left[1++\frac{y}{x}\right]}=x\frac{\left[1++\left(\frac{y}{x}\right)^2\right]}{\left[1++\frac{y}{x}\right]}=x\varphi\left(\frac{y}{x}\right)$, where ϕ is a function of $\frac{y}{x}$.

$\Rightarrow \tan u$ is a homogeneous function of degree 1 in two independent variables x, y.

Let $v=\tan u$ ......(1)

Then v is a homogeneous function of degree 1 in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v.$$

This $\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right)+y\left(\sec^2 u\frac{\partial u}{\partial y}\right)=\tan u$

$$\left(\because \frac{\partial v}{\partial x}=\sec^2 u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}\text{ and } v=\tan u,\text{ by equation (1)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}$$

$$=\frac{\sin u}{\cos u}\left(\cos^2 u\right)$$

$$=\sin u\cos u$$

$$=\frac{1}{2}(2\sin u\cos u)$$

$$=\frac{1}{2}\sin 2u.$$

Example 4 : If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{2}\cot u=0.$

Solution : Given that

$$u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

This $\Rightarrow \cos u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}=\dfrac{x\left[1++\dfrac{y}{x}\right]}{\sqrt{x}\left[1++\sqrt{\dfrac{y}{x}}\right]}=x^{\frac{1}{2}}\dfrac{\left[1++\dfrac{y}{x}\right]}{\left[1++\sqrt{\dfrac{y}{x}}\right]}=x^{\frac{1}{2}}\varphi\left(\dfrac{y}{x}\right)$, where ϕ is a function of $\dfrac{y}{x}$.

$\Rightarrow \cos u$ is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y.

Let $v=\cos u$ ......(1)

Then v is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v.$$

This $\Rightarrow x\left(-\sin u\dfrac{\partial u}{\partial x}\right)+y\left(-\sin u\dfrac{\partial u}{\partial y}\right)=\dfrac{1}{2}\cos u$

$\left(\because \dfrac{\partial v}{\partial x}=-\sin u\dfrac{\partial u}{\partial x},\ \dfrac{\partial v}{\partial y}=-\sin u\dfrac{\partial u}{\partial y} \text{ and } v=\cos u, \text{ by equation (1)}\right)$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\cos u}{\sin u}$$

$$=-\frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$

Example 5 : If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

Solution : Given that

$$u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right) \qquad ......(1)$$

Let $\sin^{-1}\left(\dfrac{x}{y}\right)=\theta.$

Then $\sin\theta=\dfrac{x}{y}.$

This $\Rightarrow \tan\theta=\dfrac{x}{\sqrt{y^2-x^2}}$

$$\Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{y^2-x^2}}$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\frac{x}{\sqrt{y^2-x^2}} \qquad .....(2)$$

Substituting the value of $\sin^{-1}\left(\frac{x}{y}\right)$ from equation (2) in equation (1), we get

$$u = \tan^{-1}\frac{x}{\sqrt{y^2-x^2}} + \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\frac{\frac{x}{\sqrt{y^2-x^2}} + \frac{y}{x}}{1-\left(\frac{x}{\sqrt{y^2-x^2}}\right)\left(\frac{y}{x}\right)}$$

$$= \tan^{-1}\frac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2} - xy}.$$

$$\text{This} \Rightarrow \tan u = \frac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2} - xy} = \frac{x^2\left[1+\frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}\right]}{x^2\left[\sqrt{\left(\frac{y}{x}\right)^2 - 1} - \frac{y}{x}\right]} = \frac{\left[1+\frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}\right]}{\left[\sqrt{\left(\frac{y}{x}\right)^2 - 1} - \frac{y}{x}\right]} = x^0\,\varphi\left(\frac{y}{x}\right),$$

where ϕ is a function of $\frac{y}{x}$.

$\Rightarrow \tan u$ is a homogeneous function of degree zero in two independent variables x, y.

Let $v = \tan u$ ......(3)

Then v is a homogeneous function of degree zero in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.$$

$$\text{This} \Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\frac{\partial u}{\partial y}\right) = 0$$

$$\left(\because \frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = \sec^2 u\frac{\partial u}{\partial y} \text{ and } v = \tan u, \text{ by equation (3)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{0}{\sec^2 u}$$

$$= 0.$$

Example 6 : If $u = \sin\left(\sqrt{x}+\sqrt{y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right)$.

Solution : Given that

$$u = \sin\left(\sqrt{x}+\sqrt{y}\right) \quad ......(1)$$

This $\Rightarrow \sin^{-1} u = \sqrt{x}+\sqrt{y} = \sqrt{x}\left[1++\sqrt{\dfrac{y}{x}}\right] = x^{\frac{1}{2}}\varphi\left(\dfrac{y}{x}\right)$, where ϕ is a function of $\dfrac{y}{x}$.

$\Rightarrow \sin^{-1} u$ is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y.

Let $v = \sin^{-1} u$ ......(2)

Then v is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v.$$

This $\Rightarrow x\left(\dfrac{1}{\sqrt{1-u^2}}\dfrac{\partial u}{\partial x}\right)+y\left(\dfrac{1}{\sqrt{1-u^2}}\dfrac{\partial u}{\partial y}\right)=\dfrac{1}{2}\sin^{-1} u$

$$\left(\because \frac{\partial v}{\partial x}=\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}=\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} \text{ and } v=\sin^{-1}u, \text{ by equation (2)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\sin^{-1}u\right)\left(\sqrt{1-u^2}\right)$$

$$=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{1-\sin^2(\sqrt{x}+\sqrt{y})}\right) \quad \text{(using equation (1))}$$

$$=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$$

Exercises

1. Verify Euler's Theorem when $u = x^3 \log\dfrac{y}{x}$.

2. If $u=\sin\sqrt{\dfrac{x-y}{x+y}}$, prove that Euler's formula $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$ holds good.

3. If $u=\sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $\dfrac{\partial u}{\partial x}=-\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

4. If $\sin u=\dfrac{x^3+y^3}{x+y}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\tan u$.

Gradient vector

- The gradient of $f(x, y, z)$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$. This gives the direction of most rapid increase at each point and the rate of change in that direction is $||\nabla f||$.
- The direction of most rapid decrease is given by $-\nabla f$ and the rate of change in that direction is $-||\nabla f||$.

Tangent plane and normal line

- The tangent plane to the graph of $z = f(x, y)$ at the point (x_0, y_0, z_0) is the plane

$$z - f(x_0, y_0) = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0).$$

- The normal line to the graph of $z = f(x, y)$ at the point (x_0, y_0, z_0) has direction

$$\boldsymbol{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle .$$

Flux and surface integrals

- The flux of the vector field $\boldsymbol{F}(x, y, z)$ through a surface σ in $\mathbb{R}^3$ is given by

$$\text{Flux} = \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n} \, dS,$$

where $\boldsymbol{n}$ is the unit normal vector depending on the orientation of the surface. If σ is the graph of $z = f(x, y)$ oriented upwards, then $\boldsymbol{n} \, dS = \langle -f_x, -f_y, 1 \rangle \, dx \, dy$.

- The surface integral of a function $H(x, y, z)$ over the graph of $z = f(x, y)$ is given by

$$\iint_\sigma H(x, y, z) \, dS = \iint_R H(x, y, f(x, y)) \cdot \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy,$$

where σ denotes the graph of $z = f(x, y)$ and R is its projection onto the xy-plane.

Change of variables

- **Cylindrical coordinates.** These are defined by the formulas

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad x^2 + y^2 = r^2, \qquad dV = r \, dz \, dr \, d\theta.$$

- **Spherical coordinates.** These are defined by the formulas

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi, \qquad dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$